Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book C

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Additional exercises for Unit C1

Section 1

Elementary operations have been used to solve the systems of equations in the following additional exercises on Section 1. If you are using these additional exercises for revision, then you may prefer to use the methods of Section 2. (Of course, you should obtain the same final answers by either method.)

Additional Exercise C1

Find the solution set of each of the following systems of linear equations.

(a)
$$\begin{aligned} x + 4y &= -7 \\ 2x - y &= 4 \\ -x + 2y &= -5 \end{aligned}$$

(b)
$$\begin{aligned} 4x - 6y &= -2 \\ -6x + 9y &= -3 \end{aligned}$$

(c)
$$\begin{aligned} p + q + r &= 5 \\ p + 2q + 3r &= 11 \\ 3p + q + 4r &= 13 \end{aligned}$$

Additional Exercise C2

Find the points of intersection of the planes $x + y - z = 0$, $y - 2z = 0$ and $3x - y + 5z = 0$ in $\mathbb{R}^3$.

Additional Exercise C3

Solve the following system of linear equations in four unknowns.

$$\begin{aligned} a - b - 2c + d &= 3 \\ b + c + d &= 3 \\ a - b - c + 2d &= 7 \\ b + c + 2d &= 7 \end{aligned}$$

Additional Exercise C4

Set up a system of linear equations for each of the following problems, then solve each system.

(a) The length of a rectangle is three times its width. The perimeter is 40 cm. Find its dimensions.

(b) The difference of two numbers is 3, and the sum of three times the larger one and twice the smaller one is 29. Find the two numbers.

Section 2

Additional Exercise C5

Which of the following are row-reduced matrices?

(a) $\begin{pmatrix} 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 14 & 0 & 23 \\ 0 & 0 & 1 & 44 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 & -3 & 12 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 & \frac{1}{7} & 0 & -6 & 24 & -\frac{3}{7} \\ 0 & 0 & 0 & 1 & 3 & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Additional Exercise C6

Solve the systems of linear equations corresponding to the following row-reduced augmented matrices. (Assume that the unknowns are $x_1, x_2, \ldots$.)

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right)$

(b) $\left(\begin{array}{ccc|c} 1 & \frac{1}{7} & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(c) $\left(\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

(d) $\left(\begin{array}{ccccc|c} 1 & 3 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

(e) $\left(\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 4 \\ 0 & 1 & -7 & 3 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Additional Exercise C7

Solve each of the following systems of linear equations by reducing its augmented matrix to row-reduced form.

(a) $$\begin{aligned} 3x - 11y - 3z &= 3 \\ 2x - 6y - 2z &= 1 \\ 5x - 17y - 6z &= 2 \\ 4x - 8y \phantom{{}- 6z} &= 7 \end{aligned}$$

(b) $$\begin{aligned} a \phantom{{}+ 2b} - 4c - 2d &= -1 \\ a + 2b - 2c + 4d &= 6 \\ 2a + 4b - 3c + 9d &= 9 \\ 2a + b - 5c + d &= -4 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 + 2x_2 - 5x_3 + 6x_4 + 10x_5 &= -2 \\ 2x_1 + 2x_2 - 6x_3 + 6x_4 + 8x_5 &= 0 \\ 2x_1 \phantom{{}+ 2x_2} - x_3 + 2x_4 + 7x_5 &= 7 \\ x_1 + 2x_2 - 5x_3 + 5x_4 + 4x_5 &= -3 \end{aligned}$$

Section 3

Additional Exercise C8

Let $\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 3 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -3 \\ 0 & 7 \end{pmatrix}$. Evaluate the following.

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - \mathbf{B}$ (c) $\mathbf{B} - \mathbf{A}$

(d) $\mathbf{AB}$ (e) $\mathbf{BA}$ (f) $\mathbf{A}^2$ (g) $\mathbf{A}^T$

(h) $\mathbf{B}^T$ (i) $\mathbf{A}^T\mathbf{B}^T$

Additional Exercise C9

Suppose that we are given matrices of the following sizes:

$\mathbf{A}$: 2×1, $\mathbf{B}$: 4×3, $\mathbf{C}$: 3×2, $\mathbf{D}$: 1×4,
$\mathbf{E}$: 3×3, $\mathbf{F}$: 3×4, $\mathbf{G}$: 2×4.

Which of the following expressions are defined? Give the size of the resulting matrix for those that are defined.

(a) $\mathbf{FB} + \mathbf{E}$ (b) $\mathbf{GF}^T - \mathbf{ADB}$

(c) $\mathbf{BF} - (\mathbf{FB})^T$ (d) $\mathbf{C}(\mathbf{AD} + \mathbf{G})$

(e) $(\mathbf{CA})^T\mathbf{D}$ (f) $\mathbf{E}^T\mathbf{B}^T\mathbf{G}^T\mathbf{C}^T$

Section 4

Additional Exercise C10

Let $\mathbf{A}$ be an invertible matrix. Prove that if k is a non-zero number, then $k\mathbf{A}$ is invertible, with $(k\mathbf{A})^{-1} = (1/k)\mathbf{A}^{-1}$.

Additional Exercise C11

Use Strategy C3 to determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 4 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 4 & 1 \\ 1 & 6 & 3 \\ 2 & 3 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}$

Additional Exercise C12

Use your answers to Additional Exercise C11(a) and (d) to solve the following systems of linear equations.

(a) $$\begin{aligned} 2x + 3y &= 3 \\ 3x + 5y &= 4 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 4x_2 + x_3 &= 4 \\ x_1 + 6x_2 + 3x_3 &= 6 \\ 2x_1 + 3x_2 \phantom{{}+ 3x_3} &= 9 \end{aligned}$$

Additional Exercise C13 Challenging

(a) Prove that a square matrix is invertible if and only if it can be expressed as a product of elementary matrices.

Hint: Use ideas from the proof of the Invertibility Theorem (Theorem C7).

(b) Express the matrix in Additional Exercise C11(a) as a product of elementary matrices.

Section 5

Additional Exercise C14

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -2 & 5 \end{pmatrix}$. Evaluate the following.

(a) $\det \mathbf{A}$ (b) $\det \mathbf{B}$ (c) $\det(\mathbf{A}+\mathbf{B})$

(d) $\det(\mathbf{AB})$ (e) $\det(\mathbf{BA})$ (f) $\det(\mathbf{A}^2)$

(g) $\det \mathbf{A}^T$ (h) $\det(\mathbf{AB})^T$ (i) $\det \mathbf{A}^{-1}$

Additional Exercise C15

Determine whether or not each of the following matrices is invertible, and find the inverse where it exists.

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 10 & 21 \\ -4 & -7 \end{pmatrix}$

Additional Exercise C16

Evaluate the determinant of each of the following matrices.

(a) $\begin{pmatrix} 5 & -31 & 10 & 12 \\ -1 & 4 & -2 & 4 \\ 2 & 10 & 4 & 16 \\ 3 & 17 & 6 & 21 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 10 & -1 & 0 & 2 \\ 1 & 5 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 3 & -6 & 3 & 7 & 1 \\ 3 & 4 & 3 & 0 & 1 \end{pmatrix}$

Additional Exercise C17

Determine whether or not each of the following matrices is invertible.

(a) $\begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 3 \\ -2 & 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

Additional Exercise C18

Prove that an $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{AA}^T$ is invertible.

Solutions to additional exercises for Unit C1

Solution to Additional Exercise C1

(a) We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + 4y = -7 \\ \mathbf{r}_2 & 2x - y = 4 \\ \mathbf{r}_3 & -x + 2y = -5 \end{array}$$

$$\begin{array}{ll} & x + 4y = -7 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & -9y = 18 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1 & 6y = -12 \end{array}$$

$$\begin{array}{ll} & x + 4y = -7 \\ \mathbf{r}_2 \to -\frac{1}{9}\mathbf{r}_2 & y = -2 \\ & 6y = -12 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_2 & x = 1 \\ & y = -2 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 6\mathbf{r}_2 & 0 = 0 \end{array}$$

We conclude that there is a unique solution: $x = 1$, $y = -2$.

(b) We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & 4x - 6y = -2 \\ \mathbf{r}_2 & -6x + 9y = -3 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \frac{1}{4}\mathbf{r}_1 & x - \frac{3}{2}y = -\frac{1}{2} \\ & -6x + 9y = -3 \end{array}$$

$$\begin{array}{ll} & x - \frac{3}{2}y = -\frac{1}{2} \\ \mathbf{r}_2 \to \mathbf{r}_2 + 6\mathbf{r}_1 & 0 = -6 \end{array}$$

Equation $\mathbf{r}_2$ is $0 = -6$, so we conclude that this system of linear equations is inconsistent: the solution set is the empty set.

(c) We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & p + q + r = 5 \\ \mathbf{r}_2 & p + 2q + 3r = 11 \\ \mathbf{r}_3 & 3p + q + 4r = 13 \end{array}$$

$$\begin{array}{ll} & p + q + r = 5 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 & q + 2r = 6 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1 & -2q + r = -2 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & p - r = -1 \\ & q + 2r = 6 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2 & 5r = 10 \end{array}$$

$$\begin{array}{ll} & p - r = -1 \\ & q + 2r = 6 \\ \mathbf{r}_3 \to \frac{1}{5}\mathbf{r}_3 & r = 2 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_3 & p = 1 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_3 & q = 2 \\ & r = 2 \end{array}$$

We conclude that there is a unique solution: $p = 1$, $q = 2$, $r = 2$.

Solution to Additional Exercise C2

The points of intersection of the three planes correspond to the solutions of the following homogeneous system of linear equations in the three unknowns x, y and z:

$$\begin{aligned} x + y - z &= 0 \\ y - 2z &= 0 \\ 3x - y + 5z &= 0. \end{aligned}$$

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y - z = 0 \\ \mathbf{r}_2 & y - 2z = 0 \\ \mathbf{r}_3 & 3x - y + 5z = 0 \end{array}$$

$$\begin{array}{ll} & x + y - z = 0 \\ & y - 2z = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1 & -4y + 8z = 0 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x + z = 0 \\ & y - 2z = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2 & 0 = 0 \end{array}$$

The $\mathbf{r}_3$ equation $(0 = 0)$ gives no constraints on x, y and z. We cannot simplify the system further: we conclude that there are infinitely many solutions.

As both equations involve a z-term, we set z equal to the real parameter k, giving

$$x = -k, \quad y = 2k, \quad z = k, \quad k \in \mathbb{R}.$$

The solution set is

$$\{(-k, 2k, k) : k \in \mathbb{R}\},$$

and so the three planes intersect in a line.

Solution to Additional Exercise C3

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll}
\mathbf{r}_1 & a-b-2c+d=3\\
\mathbf{r}_2 & b+c+d=3\\
\mathbf{r}_3 & a-b-c+2d=7\\
\mathbf{r}_4 & b+c+2d=7
\end{array}$$

$$\begin{array}{ll}
 & a-b-2c+d=3\\
 & b+c+d=3\\
\mathbf{r}_3\to\mathbf{r}_3-\mathbf{r}_1 & c+d=4\\
 & b+c+2d=7
\end{array}$$

$$\begin{array}{ll}
\mathbf{r}_1\to\mathbf{r}_1+\mathbf{r}_2 & a-c+2d=6\\
 & b+c+d=3\\
 & c+d=4\\
\mathbf{r}_4\to\mathbf{r}_4-\mathbf{r}_2 & d=4
\end{array}$$

$$\begin{array}{ll}
\mathbf{r}_1\to\mathbf{r}_1+\mathbf{r}_3 & a+3d=10\\
\mathbf{r}_2\to\mathbf{r}_2-\mathbf{r}_3 & b=-1\\
 & c+d=4\\
 & d=4
\end{array}$$

$$\begin{array}{ll}
\mathbf{r}_1\to\mathbf{r}_1-3\mathbf{r}_4 & a=-2\\
 & b=-1\\
\mathbf{r}_3\to\mathbf{r}_3-\mathbf{r}_4 & c=0\\
 & d=4
\end{array}$$

We conclude that there is a unique solution: $a=-2$, $b=-1$, $c=0$, $d=4$.

Solution to Additional Exercise C4

(a) Let the length of the rectangle be l and the width be w (both in cm).

The first statement can now be written as

$$l=3w,$$

and the second as

$$2l+2w=40.$$

We write these two equations in the usual form, label them and apply elementary operations.

$$\begin{array}{ll}
\mathbf{r}_1 & l-3w=0\\
\mathbf{r}_2 & 2l+2w=40
\end{array}$$

$$\begin{array}{ll}
 & l-3w=0\\
\mathbf{r}_2\to\mathbf{r}_2-2\mathbf{r}_1 & 8w=40
\end{array}$$

$$\begin{array}{ll}
 & l-3w=0\\
\mathbf{r}_2\to\frac{1}{8}\mathbf{r}_2 & w=5
\end{array}$$

$$\begin{array}{ll}
\mathbf{r}_1\to\mathbf{r}_1+3\mathbf{r}_2 & l=15\\
 & w=5
\end{array}$$

The system has a unique solution: $l=15$, $w=5$.

The answer to the problem is that the length of the rectangle is 15 cm and the width is 5 cm.

(b) Let the larger number be l and the smaller number be s.

The first statement can be written as

$$l-s=3,$$

and the second as

$$3l+2s=29.$$

We label these equations and apply elementary operations.

$$\begin{array}{ll}
\mathbf{r}_1 & l-s=3\\
\mathbf{r}_2 & 3l+2s=29
\end{array}$$

$$\begin{array}{ll}
 & l-s=3\\
\mathbf{r}_2\to\mathbf{r}_2-3\mathbf{r}_1 & 5s=20
\end{array}$$

$$\begin{array}{ll}
 & l-s=3\\
\mathbf{r}_2\to\frac{1}{5}\mathbf{r}_2 & s=4
\end{array}$$

$$\begin{array}{ll}
\mathbf{r}_1\to\mathbf{r}_1+\mathbf{r}_2 & l=7\\
 & s=4
\end{array}$$

The system has a unique solution: $l=7$, $s=4$.

The answer to the problem is that the two numbers are 7 and 4.

Solution to Additional Exercise C5

(a) Not row-reduced: it does not have property 3.

(b) Row-reduced.

(c) Row-reduced.

(d) Not row-reduced: it does not have property 4.

(e) Row-reduced.

(f) Row-reduced.

Solution to Additional Exercise C6

(a) The augmented matrix corresponds to the system

$$\begin{aligned}
x_1 \qquad\qquad &= 7\\
x_2 \qquad &= -6\\
x_3 &= 5.
\end{aligned}$$

The solution is $x_1=7$, $x_2=-6$, $x_3=5$.

(b) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 + \tfrac{1}{7}x_2 \quad &= 1 \\ x_3 &= 3, \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= 1 - \tfrac{1}{7}x_2, \\ x_3 &= 3. \end{aligned}$$

Setting $x_2 = k$ ($k \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= 1 - \tfrac{1}{7}k, \\ x_2 &= k, \\ x_3 &= 3. \end{aligned}$$

(c) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \quad + 4x_3 \quad &= 0 \\ x_2 - 3x_3 \quad &= 0 \\ x_4 &= 0, \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= -4x_3, \\ x_2 &= 3x_3, \\ x_4 &= 0. \end{aligned}$$

Setting $x_3 = k$ ($k \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= -4k, \\ x_2 &= 3k, \\ x_3 &= k, \\ x_4 &= 0. \end{aligned}$$

(d) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 + 3x_2 \quad - 2x_4 \quad &= 0 \\ x_3 + 2x_4 + x_5 &= 0 \\ 0 &= 1. \end{aligned}$$

The third equation cannot be satisfied, so there are no solutions.

(e) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \quad - 5x_3 \quad &= \ \ 4 \\ x_2 - 7x_3 + 3x_4 &= 12, \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= \ \ 4 + 5x_3, \\ x_2 &= 12 + 7x_3 - 3x_4. \end{aligned}$$

Setting $x_3 = k$ and $x_4 = l$ ($k, l \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= 4 + 5k, \\ x_2 &= 12 + 7k - 3l, \\ x_3 &= k, \\ x_4 &= l. \end{aligned}$$

Solution to Additional Exercise C7

We follow Strategy C2.

(Although your row-reduction sequences may differ from these, your final answers should agree!)

(a) We row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \left(\begin{array}{rrr|r} 3 & -11 & -3 & 3 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{array}\right) \quad \begin{array}{r} -8 \\ -5 \\ -16 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \\ \\ \end{array} \quad \left(\begin{array}{rrr|r} 1 & -5 & -1 & 2 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{array}\right) \quad \begin{array}{r} -3 \\ -5 \\ -16 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 5\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 4\mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|r} 1 & -5 & -1 & 2 \\ 0 & 4 & 0 & -3 \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{array}\right) \quad \begin{array}{r} -3 \\ 1 \\ -1 \\ 15 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{4}\mathbf{r}_2 \\ \\ \\ \end{array} \quad \left(\begin{array}{rrr|r} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{array}\right) \quad \begin{array}{r} -3 \\ \frac{1}{4} \\ -1 \\ 15 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 5\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 8\mathbf{r}_2 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 12\mathbf{r}_2 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 0 & -1 & -\frac{7}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & 8 \end{array}\right) \quad \begin{array}{r} -\frac{7}{4} \\ \frac{1}{4} \\ -3 \\ 12 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\mathbf{r}_3 \\ \\ \end{array} \quad \left(\begin{array}{rrr|r} 1 & 0 & -1 & -\frac{7}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \end{array}\right) \quad \begin{array}{r} -\frac{7}{4} \\ \frac{1}{4} \\ 3 \\ 12 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_3 \\ \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 - 4\mathbf{r}_3 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{r} \frac{5}{4} \\ \frac{1}{4} \\ 3 \\ 0 \end{array}$$

This matrix is in row-reduced form.
The corresponding system is

$$\begin{aligned} x \qquad &= \tfrac{1}{4} \\ y \quad &= -\tfrac{3}{4} \\ z &= 2. \end{aligned}$$

Thus the solution is $x = \frac{1}{4}$, $y = -\frac{3}{4}$, $z = 2$.

(b) We row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 1 & 2 & -2 & 4 & 6 \\ 2 & 4 & -3 & 9 & 9 \\ 2 & 1 & -5 & 1 & -4 \end{array}\right) \quad \begin{array}{r} -6 \\ 11 \\ 21 \\ -5 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 2\mathbf{r}_1 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 2 & 2 & 6 & 7 \\ 0 & 4 & 5 & 13 & 11 \\ 0 & 1 & 3 & 5 & -2 \end{array}\right) \quad \begin{array}{r} -6 \\ 17 \\ 33 \\ 7 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_4 \\ \\ \\ \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 1 & -1 & 1 & 9 \\ 0 & 4 & 5 & 13 & 11 \\ 0 & 1 & 3 & 5 & -2 \end{array}\right) \quad \begin{array}{r} -6 \\ 10 \\ 33 \\ 7 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 4\mathbf{r}_2 \\ \mathbf{r}_4 \to \mathbf{r}_4 - \mathbf{r}_2 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 1 & -1 & 1 & 9 \\ 0 & 0 & 9 & 9 & -25 \\ 0 & 0 & 4 & 4 & -11 \end{array}\right) \quad \begin{array}{r} -6 \\ 10 \\ -7 \\ -3 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_4 \\ \\ \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & -4 & -2 & -1 \\ 0 & 1 & -1 & 1 & 9 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 4 & 4 & -11 \end{array}\right) \quad \begin{array}{r} -6 \\ 10 \\ -1 \\ -3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 4\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 - 4\mathbf{r}_3 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -13 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{r} -10 \\ 9 \\ -1 \\ 1 \end{array}$$

This matrix is in row-reduced form.
The corresponding system is

$$\begin{aligned} a \qquad + 2d &= -13 \\ b \quad + 2d &= 6 \\ c + d &= -3 \\ 0 &= 1. \end{aligned}$$

The fourth equation cannot be satisfied, so there are no solutions.

The last matrix in the above row-reduction corresponds to a system containing the equation $0 = 1$. We could have concluded that the given system has no solutions at this point, or earlier.

(c) We row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \left(\begin{array}{ccccc|c} 2 & 2 & -5 & 6 & 10 & -2 \\ 2 & 2 & -6 & 6 & 8 & 0 \\ 2 & 0 & -1 & 2 & 7 & 7 \\ 1 & 2 & -5 & 5 & 4 & -3 \end{array}\right) \quad \begin{array}{r} 13 \\ 12 \\ 17 \\ 4 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_4 \\ \\ \\ \\ \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 5 & 4 & -3 \\ 2 & 2 & -6 & 6 & 8 & 0 \\ 2 & 0 & -1 & 2 & 7 & 7 \\ 2 & 2 & -5 & 6 & 10 & -2 \end{array}\right) \quad \begin{array}{r} 4 \\ 12 \\ 17 \\ 13 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 2\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 5 & 4 & -3 \\ 0 & -2 & 4 & -4 & 0 & 6 \\ 0 & -4 & 9 & -8 & -1 & 13 \\ 0 & -2 & 5 & -4 & 2 & 4 \end{array}\right) \quad \begin{array}{r} 4 \\ 4 \\ 9 \\ 5 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{2}\mathbf{r}_2 \\ \\ \\ \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 5 & 4 & -3 \\ 0 & 1 & -2 & 2 & 0 & -3 \\ 0 & -4 & 9 & -8 & -1 & 13 \\ 0 & -2 & 5 & -4 & 2 & 4 \end{array}\right) \quad \begin{array}{r} 4 \\ -2 \\ 9 \\ 5 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2 \\ \mathbf{r}_4 \to \mathbf{r}_4 + 2\mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & 4 & 3 \\ 0 & 1 & -2 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -2 \end{array}\right) \quad \begin{array}{r} 8 \\ -2 \\ 1 \\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 - \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 4 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{array}\right) \quad \begin{array}{r} 9 \\ 0 \\ 1 \\ 0 \end{array}$$

$$\begin{array}{l} \\ \\ \\ \mathbf{r}_4 \to \frac{1}{3}\mathbf{r}_4 \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 4 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{array}{r} 9 \\ 0 \\ 1 \\ 0 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_4 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_4 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_4 \\ \\ \end{array} \quad \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{array}{r} 9 \\ 0 \\ 1 \\ 0 \end{array}$$

This matrix is in row-reduced form.
The corresponding system is

$$\begin{aligned} x_1 \qquad + x_4 \qquad &= 7 \\ x_2 \quad + 2x_4 \qquad &= -3 \\ x_3 \qquad\qquad &= 0 \\ x_5 &= -1, \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= 7 \quad - x_4 \\ x_2 &= -3 - 2x_4 \\ x_3 &= 0 \\ x_5 &= -1. \end{aligned}$$

Setting $x_4 = k$ ($k \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= 7 - k, \\ x_2 &= -3 - 2k, \\ x_3 &= 0, \\ x_4 &= k, \\ x_5 &= -1. \end{aligned}$$

Solution to Additional Exercise C8

(a) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}3&3\\3&3\end{pmatrix}$

(b) $\mathbf{A}-\mathbf{B}=\begin{pmatrix}-1&9\\3&-11\end{pmatrix}$

(c) $\mathbf{B}-\mathbf{A}=-(\mathbf{A}-\mathbf{B})=\begin{pmatrix}1&-9\\-3&11\end{pmatrix}$

(d) $\mathbf{AB}=\begin{pmatrix}2&39\\6&-37\end{pmatrix}$

(e) $\mathbf{BA}=\begin{pmatrix}-7&24\\21&-28\end{pmatrix}$

(f) $\mathbf{A}^2=\begin{pmatrix}19&-18\\-9&34\end{pmatrix}$

(g) $\mathbf{A}^T=\begin{pmatrix}1&3\\6&-4\end{pmatrix}$

(h) $\mathbf{B}^T=\begin{pmatrix}2&0\\-3&7\end{pmatrix}$

(i) $\mathbf{A}^T\mathbf{B}^T=(\mathbf{BA})^T=\begin{pmatrix}-7&21\\24&-28\end{pmatrix}$

Solution to Additional Exercise C9

(a) $\mathbf{FB}: 3\times 3,\quad \mathbf{E}: 3\times 3.$

The matrix $\mathbf{FB}+\mathbf{E}$ exists and is 3×3.

(b) $\mathbf{F}^T: 4\times 3,\quad \mathbf{GF}^T: 2\times 3,\quad \mathbf{AD}: 2\times 4,\quad \mathbf{ADB}: 2\times 3.$

The matrix $\mathbf{GF}^T-\mathbf{ADB}$ exists and is 2×3.

(c) $\mathbf{BF}: 4\times 4,\quad \mathbf{FB}: 3\times 3,\quad (\mathbf{FB})^T: 3\times 3.$

The matrix $\mathbf{BF}-(\mathbf{FB})^T$ does not exist, since $\mathbf{BF}$ and $(\mathbf{FB})^T$ are different sizes.

(d) $\mathbf{AD}: 2\times 4,\quad \mathbf{AD}+\mathbf{G}: 2\times 4,\quad \mathbf{C}: 3\times 2.$

The matrix $\mathbf{C}(\mathbf{AD}+\mathbf{G})$ exists and is 3×4.

(e) $\mathbf{CA}: 3\times 1,\quad (\mathbf{CA})^T: 1\times 3,\quad \mathbf{D}: 1\times 4.$

The matrix $(\mathbf{CA})^T\mathbf{D}$ does not exist, since $(\mathbf{CA})^T$ is size 1×3 and $\mathbf{D}$ is size 1×4.

(f) $\mathbf{E}^T\mathbf{B}^T\mathbf{G}^T\mathbf{C}^T=(\mathbf{CGBE})^T.$

$\mathbf{CG}: 3\times 4,\quad \mathbf{CGB}: 3\times 3,\quad \mathbf{CGBE}: 3\times 3,\quad (\mathbf{CGBE})^T: 3\times 3.$

The matrix $\mathbf{E}^T\mathbf{B}^T\mathbf{G}^T\mathbf{C}^T$ exists and is 3×3.

Solution to Additional Exercise C10

To prove that $k\mathbf{A}$ is invertible, with inverse $(1/k)\mathbf{A}^{-1}$, we have to show that

$$(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right)=\mathbf{I}=\left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}).$$

But

$$\begin{aligned}(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right)&=\left(k\times\frac{1}{k}\right)(\mathbf{AA}^{-1})\\&=1\mathbf{I}=\mathbf{I},\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A})&=\left(\frac{1}{k}\times k\right)(\mathbf{A}^{-1}\mathbf{A})\\&=1\mathbf{I}=\mathbf{I},\end{aligned}$$

as required.

Solution to Additional Exercise C11

(a) We row-reduce $(\mathbf{A}\mid\mathbf{I})$.

$$\begin{array}{l}\mathbf{r}_1\\ \mathbf{r}_2\end{array}\quad\left(\begin{array}{cc|cc}2&3&1&0\\3&5&0&1\end{array}\right)\begin{array}{c}6\\9\end{array}$$

$$\mathbf{r}_1\to\tfrac{1}{2}\mathbf{r}_1\quad\left(\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\3&5&0&1\end{array}\right)\begin{array}{c}3\\9\end{array}$$

$$\mathbf{r}_2\to\mathbf{r}_2-3\mathbf{r}_1\quad\left(\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&-\frac{3}{2}&1\end{array}\right)\begin{array}{c}3\\0\end{array}$$

$$\mathbf{r}_2\to 2\mathbf{r}_2\quad\left(\begin{array}{cc|cc}1&\frac{3}{2}&\frac{1}{2}&0\\0&1&-3&2\end{array}\right)\begin{array}{c}3\\0\end{array}$$

$$\mathbf{r}_1\to\mathbf{r}_1-\tfrac{3}{2}\mathbf{r}_2\quad\left(\begin{array}{cc|cc}1&0&5&-3\\0&1&-3&2\end{array}\right)\begin{array}{c}3\\0\end{array}$$

The left half has been reduced to $\mathbf{I}$, so the given matrix is invertible: its inverse is

$$\mathbf{A}^{-1}=\begin{pmatrix}5&-3\\-3&2\end{pmatrix}.$$

(b) We row-reduce $(\mathbf{A}\mid\mathbf{I})$.

$$\begin{array}{l}\mathbf{r}_1\\ \mathbf{r}_2\end{array}\quad\left(\begin{array}{cc|cc}-2&4&1&0\\3&-6&0&1\end{array}\right)\begin{array}{c}3\\-2\end{array}$$

$$\mathbf{r}_1\to-\tfrac{1}{2}\mathbf{r}_1\quad\left(\begin{array}{cc|cc}1&-2&-\frac{1}{2}&0\\3&-6&0&1\end{array}\right)\begin{array}{c}-\frac{3}{2}\\-2\end{array}$$

$$\mathbf{r}_2\to\mathbf{r}_2-3\mathbf{r}_1\quad\left(\begin{array}{cc|cc}1&-2&-\frac{1}{2}&0\\0&0&\frac{3}{2}&1\end{array}\right)\begin{array}{c}-\frac{3}{2}\\\frac{5}{2}\end{array}$$

The left half is now in row-reduced form, but it is not the identity matrix. Therefore the given matrix is not invertible.

(c) We row-reduce $(\mathbf{A}\mid\mathbf{I})$.

$$\begin{array}{l}\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\end{array}\quad\left(\begin{array}{ccc|ccc}1&2&4&1&0&0\\-2&-1&1&0&1&0\\1&1&1&0&0&1\end{array}\right)\begin{array}{c}8\\-1\\4\end{array}$$

$$\begin{array}{l}\\ \mathbf{r}_2\to\mathbf{r}_2+2\mathbf{r}_1\\ \mathbf{r}_3\to\mathbf{r}_3-\mathbf{r}_1\end{array}\quad\left(\begin{array}{ccc|ccc}1&2&4&1&0&0\\0&3&9&2&1&0\\0&-1&-3&-1&0&1\end{array}\right)\begin{array}{c}8\\15\\-4\end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \tfrac{1}{3}\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right) \begin{array}{r} 8 \\ 5 \\ -4 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \begin{array}{r} -2 \\ 5 \\ 1 \end{array}$$

The left half is now in row-reduced form, but it is not the identity matrix. Therefore the given matrix is not invertible.

(d) We row-reduce $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 1 & 6 & 3 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 7 \\ 11 \\ 6 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & -5 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{r} 7 \\ 4 \\ -8 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \tfrac{1}{2}\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -5 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{r} 7 \\ 2 \\ -8 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 5\mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & -\frac{9}{2} & \frac{5}{2} & 1 \end{array}\right) \begin{array}{r} -1 \\ 2 \\ 2 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \tfrac{1}{3}\mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{array}\right) \begin{array}{r} -1 \\ 2 \\ \frac{2}{3} \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 3\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{array}\right) \begin{array}{r} 1 \\ \frac{4}{3} \\ \frac{2}{3} \end{array}$$

The left half has been reduced to $\mathbf{I}$, so the given matrix is invertible: its inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{pmatrix}.$$

(e) We row-reduce $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ 4 \\ -1 \\ -2 \end{array}$$

$$\begin{array}{l} \\ \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 + \mathbf{r}_1 \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ 4 \\ -1 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ 4 \\ 3 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_3 \\ \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ -2 \\ 3 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_4 \\ \\ \\ \\ \end{array} \quad \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} -4 \\ -2 \\ 3 \\ 3 \end{array}$$

The left half has been reduced to $\mathbf{I}$, so the given matrix is invertible: its inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} -2 & 0 & 0 & -3 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Solution to Additional Exercise C12

(a) The matrix form of the system is

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Multiplying this equation on the left by the inverse of the coefficient matrix (from Additional Exercise C11(a)) gives the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix};$$

that is, $x = 3$, $y = -1$.

(b) The matrix form of the system is

$$\begin{pmatrix} 1 & 4 & 1 \\ 1 & 6 & 3 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}.$$

Multiplying this equation on the left by the inverse of the coefficient matrix (from Additional Exercise C11(d)) gives the solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix};$$

that is, $x_1 = 6$, $x_2 = -1$, $x_3 = 2$.

Solution to Additional Exercise C13

(a) Let $\mathbf{A}$ be an $n \times n$ matrix.

First we show that if a square matrix can be expressed as a product of elementary matrices, then it is invertible.

Every elementary matrix is invertible (by Corollary C13), and the product of invertible matrices is invertible (by Theorem C5). So if $\mathbf{A}$

can be expressed as a product of elementary matrices, then $\mathbf{A}$ is invertible.

Next we show that if a square matrix is invertible, then it can be expressed as a product of elementary matrices.

Suppose that $\mathbf{A}$ is invertible. Then, by the Invertibility Theorem, the row-reduced form of $\mathbf{A}$ is $\mathbf{I}$. Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $n \times n$ elementary matrices associated with a sequence of elementary row operations that transforms $\mathbf{A}$ to $\mathbf{I}$, in the same order. Then, by Corollary C11,

$$\mathbf{I} = \mathbf{BA},$$

where $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$. Now $\mathbf{B}$ is a product of invertible matrices, and is therefore itself an invertible matrix. Multiplying both sides of the above equation on the left by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}\mathbf{BA},$$

that is,

$$\mathbf{B}^{-1} = \mathbf{A}.$$

So

$$(\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{A},$$

which, by Theorem C5, is equivalent to

$$\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\cdots\mathbf{E}_k^{-1} = \mathbf{A}.$$

By Corollary C13, the inverse of every elementary matrix is an elementary matrix, and so this expresses $\mathbf{A}$ as a product of elementary matrices.

(b) Let $\mathbf{A}$ be the matrix in Additional Exercise C11(a). It is transformed to $\mathbf{I}$ by the sequence of elementary row operations $\mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1$, $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$, $\mathbf{r}_2 \to 2\mathbf{r}_2$, $\mathbf{r}_1 \to \mathbf{r}_1 - \frac{3}{2}\mathbf{r}_2$ (see the solution to Additional Exercise C11(a)), with associated elementary matrices

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}.$$

Therefore, by the argument in part (a) above,

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

(The inverses of the elementary matrices are easily found using the argument in the proof of Corollary C13.)

Using a different sequence of row operations to row-reduce $\mathbf{A}$ would lead to a different way of expressing $\mathbf{A}$ as a product of elementary matrices – so your solution may differ from the one above.

Solution to Additional Exercise C14

(a) $\det \mathbf{A} = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = (2 \times 1) - (0 \times 4) = 2$

(b) $\det \mathbf{B} = \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} = (1 \times 5) - (-1 \times (-2)) = 3$

(c)

$\det(\mathbf{A}+\mathbf{B}) = \begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} = (3 \times 6) - (-1 \times 2) = 20$

(d) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}) = 6$

(e) $\det(\mathbf{BA}) = (\det \mathbf{B})(\det \mathbf{A}) = 6$

(f) $\det(\mathbf{A}^2) = (\det \mathbf{A})^2 = 4$

(g) $\det \mathbf{A}^T = \det \mathbf{A} = 2$

(h) $\det(\mathbf{AB})^T = \det(\mathbf{AB}) = 6$

(i) Since $\det \mathbf{A} \neq 0$,

$$\det \mathbf{A}^{-1} = 1/(\det \mathbf{A}) = \tfrac{1}{2}.$$

Solution to Additional Exercise C15

(a) We first evaluate the determinant of the matrix:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

This determinant is non-zero, so the matrix is invertible by Theorem C17. The inverse is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

(b) We first evaluate the determinant of the matrix:

$$\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0.$$

This determinant is zero, so the matrix is not invertible by Theorem C17.

(c) We first evaluate the determinant of the matrix:

$$\begin{vmatrix} 10 & 21 \\ -4 & -7 \end{vmatrix} = 14.$$

This determinant is non-zero, so the matrix is invertible by Theorem C17. The inverse is

$$\begin{aligned} \begin{pmatrix} 10 & 21 \\ -4 & -7 \end{pmatrix}^{-1} &= \tfrac{1}{14}\begin{pmatrix} -7 & -21 \\ 4 & 10 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix}. \end{aligned}$$

Solution to Additional Exercise C16

(a) The first and third columns are proportional, since

$$2\begin{pmatrix} 5 \\ -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 4 \\ 6 \end{pmatrix}.$$

The determinant is therefore zero, by Theorem C16.

(b) We interchange the first and third rows, and apply Theorems C14 and C15, giving

$$\begin{aligned} &\begin{vmatrix} 7 & 10 & -1 & 0 & 2 \\ 1 & 5 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 3 & -6 & 3 & 7 & 1 \\ 3 & 4 & 3 & 0 & 1 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 5 & 0 & 2 & 0 \\ 7 & 10 & -1 & 0 & 2 \\ 3 & -6 & 3 & 7 & 1 \\ 3 & 4 & 3 & 0 & 1 \end{vmatrix} \\ &= (-1)(-2)\begin{vmatrix} 1 & 0 & 2 & 0 \\ 7 & -1 & 0 & 2 \\ 3 & 3 & 7 & 1 \\ 3 & 3 & 0 & 1 \end{vmatrix} \\ &= 2\left(\begin{vmatrix} -1 & 0 & 2 \\ 3 & 7 & 1 \\ 3 & 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 7 & -1 & 2 \\ 3 & 3 & 1 \\ 3 & 3 & 1 \end{vmatrix}\right) \\ &= 2\left(-1\begin{vmatrix} 7 & 1 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 7 \\ 3 & 0 \end{vmatrix} + 0\right) \\ &= 2(-7 - 42) \\ &= -98. \end{aligned}$$

Notice that the second 3×3 determinant vanishes, by Theorem C16, since the second and third rows are equal.

Solution to Additional Exercise C17

In each part, we evaluate the determinant to determine whether or not the matrix is invertible.

(a) We have

$$\begin{aligned} &\begin{vmatrix} 2 & 1 & 4 \\ 1 & -1 & 3 \\ -2 & 1 & 5 \end{vmatrix} \\ &\qquad = 2\begin{vmatrix} -1 & 3 \\ 1 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} \\ &\qquad = -16 - 11 - 4 \\ &\qquad = -31. \end{aligned}$$

The determinant of this matrix is non-zero, so the matrix is invertible by Theorem C17.

(b) By Theorem C16, the determinant is zero, since the first and third columns are equal. The matrix is therefore not invertible by Theorem C17.

(c) We have

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1.$$

The determinant is non-zero, so the matrix is invertible.

Solution to Additional Exercise C18

We first show that if $\mathbf{A}\mathbf{A}^T$ is invertible, then $\mathbf{A}$ is invertible.

Suppose that $\mathbf{A}\mathbf{A}^T$ is invertible. Then, by Theorem C17, $\det(\mathbf{A}\mathbf{A}^T)$ is non-zero, and by Theorem C14

$$\det(\mathbf{A}\mathbf{A}^T) = (\det \mathbf{A})(\det \mathbf{A}^T).$$

So $\det \mathbf{A}$ and $\det \mathbf{A}^T$ are also non-zero, and therefore $\mathbf{A}$ and $\mathbf{A}^T$ are both invertible.

We now show that if $\mathbf{A}$ is invertible, then $\mathbf{A}\mathbf{A}^T$ is invertible.

Suppose that $\mathbf{A}$ is invertible. Then $\det \mathbf{A}$ and $\det \mathbf{A}^T$ are non-zero, by Theorems C14 and C17, so

$$(\det \mathbf{A})(\det \mathbf{A}^T) = \det(\mathbf{A}\mathbf{A}^T) \neq 0.$$

So $\mathbf{A}\mathbf{A}^T$ is invertible, as required.

Additional exercises for Unit C2

Section 1

Additional Exercise C19

Show that the following are not vector spaces, by finding an axiom that fails. Assume the usual definitions of addition and scalar multiplication for the elements of these sets.

(a) $V = \{2 + ax : a \in \mathbb{R}\}$

(b) $V = \{(x, y, x + y - 3) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

(c) $V = \{(x, y) : x \geq 0,\ x, y \in \mathbb{R}\}$

(d) $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$

Section 2

Additional Exercise C20

Calculate $2\mathbf{u} - \mathbf{v}$ for each of the following vectors $\mathbf{u}$ and $\mathbf{v}$, and draw a diagram in part (a) showing the resulting vector geometrically.

(a) In $\mathbb{R}^2$, $\mathbf{u} = (3, -1)$ and $\mathbf{v} = (2, 4)$.

(b) In $\mathbb{R}^3$, $\mathbf{u} = (1, 2, 0)$ and $\mathbf{v} = \left(0, -1, \frac{3}{4}\right)$.

Additional Exercise C21

Let $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (-1, 1, 0)$, $\mathbf{w} = (0, 0, 1)$ and $\mathbf{z} = (1, 1, 0)$.

(a) Calculate the linear combination $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$.

(b) Find real numbers α, β, γ such that

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = (0, 1, 0).$$

(c) Do there exist real numbers α and β such that

$$\alpha\mathbf{u} + \beta\mathbf{v} = (0, 1, 0)?$$

(d) Find all possible real numbers α, β, γ and δ such that

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} + \delta\mathbf{z} = (0, 1, 0).$$

Additional Exercise C22

Let $S = \{(1, 1, 0), (0, 1, 1)\}$.

(a) Do the vectors $(0, 2, 0)$ and $(5, 1, -4)$ belong to the span $\langle S \rangle$?

(b) Give a simple geometric description of the span $\langle S \rangle$.

Additional Exercise C23

Give a simple geometric description of the span $\langle S \rangle$ of the given sets.

(a) In $\mathbb{R}^2$, $S = \{(2, -1)\}$.

(b) In $\mathbb{R}^3$, $S = \{(0, -1, 1), (0, 2, -3)\}$.

Additional Exercise C24

Give an algebraic description of the span $\langle S \rangle$ of the given sets.

(a) In P_3, the set $S = \{1 + 2x\}$.

(b) In $M_{2,2}$, the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$.

Additional Exercise C25

Show that the set $\{(-1, 0), (2, 1)\}$ spans $\mathbb{R}^2$.

Section 3

Additional Exercise C26

Determine which of the following sets are linearly independent.

(a) $\{(0, 0), (1, 1)\}$ in $\mathbb{R}^2$.

(b) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $\mathbb{R}^3$.

(c) $\{1 + 2x, 3x, 2 - 4x\}$ in P_2.

Additional Exercise C27

By testing for linear independence and spanning, using Strategy C8, determine in each case whether the set S is a basis for the vector space V.

(a) $V = M_{2,1}, \quad S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$.

(b) $V = P_3, \quad S = \{2 + x, 3 - x^2\}$.

(c) $V = \mathbb{C}, \quad S = \{1 - 2i, 2 - 4i\}$.

Additional Exercise C28

Determine in each of the following cases whether the set S is a basis for the vector space V.

(a) $V = \mathbb{R}^2, \quad S = \left\{ \left(\frac{1}{2}, 2\right), \left(2, \frac{1}{2}\right) \right\}$.

(b) $V = \mathbb{R}^3, \quad S = \{(1, 2, -3), (4, 7, -5)\}$.

(c) $V = \mathbb{R}^4, \quad S = \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$.

Additional Exercise C29

(a) Let $E = \{(-3, 1), (1, 2)\}$ be a basis for $\mathbb{R}^2$. Determine the standard coordinate representation of $(2, 1)_E$.

(b) Let $E = \{(1, 0, 2), (-1, 1, 3), (2, -2, 0)\}$ be a basis for $\mathbb{R}^3$. Determine the standard coordinate representation of $(1, -2, 3)_E$.

Additional Exercise C30

(a) Find the E-coordinate representation of the vector $(6, 5)$ with respect to the basis $E = \{(-3, 1), (1, 2)\}$ for $\mathbb{R}^2$.

(b) Find the E-coordinate representation of the vector $(3, 5, -5)$ with respect to the basis $E = \{(1, 2, 0), (-1, 3, 1), (0, 2, -2)\}$ for $\mathbb{R}^3$.

Section 4

Additional Exercise C31

Prove Theorem C27 by considering each of the axioms in the definition of a vector space and showing that they are true for a subset of a vector space if the conditions of Theorem C27 are satisfied.

Theorem C27

A subset S of a vector space V is a subspace of V if it satisfies the following conditions.

(a) $\mathbf{0} \in S$.

(b) S is closed under vector addition.

(c) S is closed under scalar multiplication.

Additional Exercise C32

Determine whether each of the following subsets S is a subspace of the given vector space V.

(a) $V = \mathbb{R}^3, \quad S = \{(x, y, 2x + y) : x, y \in \mathbb{R}\}$.

(b) $V = \mathbb{R}^2, \quad S = \{(x, x - 3) : x \in \mathbb{R}\}$.

(c) $V = \mathbb{R}^4$,
$S = \{(x, y, x + 3y, 2x - y) : x, y \in \mathbb{R}\}$.

(d) $V = P_3, \quad S = \{ax^2 : a \in \mathbb{R}\}$.

(e) $V = \mathbb{R}^2$,
$S = \{(x, y) : x \geq 0, \ y \geq 0, \ x, y \in \mathbb{R}\}$.

(f) $V = M_{3,1}, \quad S = \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\} \right\rangle$.

Additional Exercise C33

Find a basis, and hence the dimension, for each of the subspaces in Additional Exercise C32.

Section 5

Additional Exercise C34

Let $\mathbf{v}_1 = (1, 5, -3, 4, -7)$ and $\mathbf{v}_2 = (2, 8, 0, -7, -2)$ be vectors in $\mathbb{R}^5$.

(a) Calculate the scalar product $\mathbf{v}_1 \cdot \mathbf{v}_2$.

(b) Determine the magnitude of $\mathbf{v}_1$ and of $\mathbf{v}_2$.

Additional Exercise C35

(a) Verify that

$$\{(5, 5, 5, 5), (5, -5, -5, 5), (5, 0, 0, -5), (0, 5, -5, 0)\}$$

is an orthogonal basis for $\mathbb{R}^4$.

(b) Express the vector $(5, 0, 0, 0)$ as a linear combination of the basis vectors in part (a).

(c) Determine the corresponding orthonormal basis for $\mathbb{R}^4$.

Additional Exercise C36

(a) Choose a pair of linearly independent vectors that lie in the plane orthogonal to $(1, -2, 2)$.

Hint: Choose vectors that are as simple as possible to work with, for example, ones containing small numbers and at least one zero.

(b) Apply the Gram–Schmidt orthogonalisation process to find an orthogonal basis for the plane orthogonal to $(1, -2, 2)$.

(c) Hence write down an orthogonal basis for $\mathbb{R}^3$ containing the vector $(1, -2, 2)$.

(d) Determine the corresponding orthonormal basis for $\mathbb{R}^3$.

Additional Exercise C37

Starting with the basis

$$\{(2, 2, 1, 0), (1, 2, 0, 2), (0, 1, 2, 2), (2, 0, 2, 1)\}$$

for $\mathbb{R}^4$, use the Gram–Schmidt orthogonalisation process to find an orthogonal basis for $\mathbb{R}^4$.

Solutions to additional exercises for Unit C2

Solution to Additional Exercise C19

(a) Consider $p_1(x) = 2$ and $p_2(x) = 2 + x$, which belong to V. Then

$$\begin{aligned} p_1(x) + p_2(x) &= 2 + (2 + x) \\ &= 4 + x. \end{aligned}$$

This polynomial is not of the form $p(x) = 2 + ax$ and so does not belong to V.

Therefore V fails to satisfy the closure axiom (A1), and so is not a vector space.

Other axioms also fail here, or do not make sense: for example, you may have spotted that V fails the additive identity axiom (A3) since it contains no zero vector, and therefore the additive inverses property (A4) makes no sense here.

Alternatively, you may have spotted that the closure axiom (S1) also fails: let $\alpha \in \mathbb{R}$, then $\alpha p_1(x) = 2\alpha$, which is not of the form $p(x) = 2 + ax$ and so does not belong to V.

(b) Consider $\mathbf{u} = (1, 0, -2)$ and $\mathbf{v} = (0, 1, -2)$, which belong to V. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (1, 0, -2) + (0, 1, -2) \\ &= (1, 1, -4). \end{aligned}$$

This does not belong to V, since

$$1 + 1 - 3 = -1 \neq -4.$$

Therefore V fails to satisfy the closure axiom (A1), and so is not a vector space.

Again, other axioms also fail here: for example, you may have spotted that V fails the additive identity axiom (A3) since it contains no zero vector, and it fails the closure axiom (S1).

(c) Consider $\mathbf{u} = (1, 0)$ and $\alpha = -1$. Then $\mathbf{u} \in V$, but $-1\mathbf{u} = (-1, 0)$ does not belong to V, since $-1 < 0$.

Therefore V fails to satisfy the closure axiom (S1).

Again, other axioms fail here, although the closure axiom (A1) and additive identity axiom (A3) do hold. You may have spotted that the additive inverses property (A4) fails because $-\mathbf{u} = (-1, 0)$ is not in V.

(d) Consider the following matrices with determinant 1 (that is, for each, $ad - bc = 1$):

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathbf{A}, \mathbf{B} \in V$, but

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

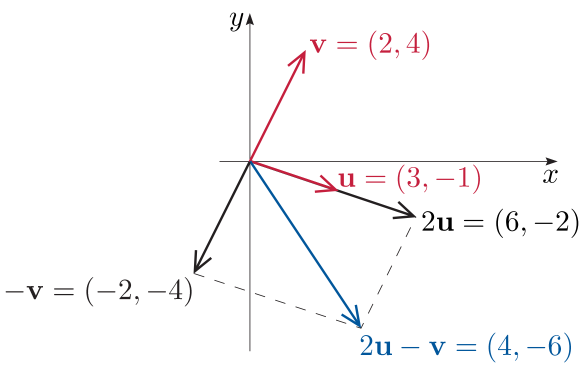
has determinant $(1 \times 1) - (-1 \times 1) = 2$, and so does not belong to V.

Therefore V fails to satisfy the closure axiom (A1).

Again, other axioms also fail here, for example, you may have spotted that V fails the additive identity axiom (A3) since it contains no zero vector, or it fails the closure axiom (S1).

Solution to Additional Exercise C20

(a)
$$\begin{aligned} 2\mathbf{u} - \mathbf{v} &= 2(3, -1) - (2, 4) \\ &= (6, -2) - (2, 4) = (4, -6) \end{aligned}$$



(b)
$$\begin{aligned} 2\mathbf{u} - \mathbf{v} &= (2, 4, 0) - \left(0, -1, \tfrac{3}{4}\right) \\ &= \left(2, 5, -\tfrac{3}{4}\right) \end{aligned}$$

Solution to Additional Exercise C21

(a)
$$\begin{aligned} &2\mathbf{u} - \mathbf{v} + 3\mathbf{w} \\ &\quad = 2(1, 0, 1) - (-1, 1, 0) + 3(0, 0, 1) \\ &\quad = (2, 0, 2) - (-1, 1, 0) + (0, 0, 3) \\ &\quad = (3, -1, 5) \end{aligned}$$

(b)
$$\begin{aligned} &\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \\ &\quad = \alpha(1, 0, 1) + \beta(-1, 1, 0) + \gamma(0, 0, 1) \\ &\quad = (\alpha - \beta, \beta, \alpha + \gamma) = (0, 1, 0) \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta \quad\quad &= 0 \\ \beta \quad\quad &= 1 \\ \alpha \quad + \gamma &= 0. \end{aligned}$$

Solving these equations gives $\alpha = \beta = 1$ and $\gamma = -1$.

(c) From part (b), the only possibility is $\alpha = \beta = 1$. But

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (1,0,1) + (-1,1,0) \\ &= (0,1,1) \neq (0,1,0). \end{aligned}$$

So there do not exist any suitable α and β.

(d) $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} + \delta\mathbf{z}$

$$\begin{aligned} &= \alpha(1,0,1) + \beta(-1,1,0) + \gamma(0,0,1) \\ &\quad + \delta(1,1,0) \\ &= (\alpha - \beta + \delta, \beta + \delta, \alpha + \gamma) \\ &= (0,1,0). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta \quad\quad + \delta &= 0 \\ \beta \quad\quad + \delta &= 1 \\ \alpha \quad + \gamma \quad\quad &= 0. \end{aligned}$$

Solving these equations, we let δ be any real number k. Then we find

$$\beta = 1 - k, \ \ \alpha = 1 - 2k, \ \ \gamma = -1 + 2k.$$

The general solution is

$$\begin{aligned} \alpha &= 1 - 2k, \\ \beta &= 1 - k, \\ \gamma &= -1 + 2k, \\ \delta &= k, \end{aligned}$$

for any real number k.

(There are other forms of this solution if you let one of the other coefficients equal k, but they are all equivalent.)

Solution to Additional Exercise C22

(a) We use Strategy C6.

We write

$$\begin{aligned} (0,2,0) &= \alpha(1,1,0) + \beta(0,1,1) \\ &= (\alpha, \alpha + \beta, \beta). \end{aligned}$$

Equating corresponding coordinates gives the system

$$\begin{aligned} \alpha \quad\quad &= 0 \\ \alpha + \beta &= 2 \\ \beta &= 0. \end{aligned}$$

This system is inconsistent and therefore has no solution, so $(0,2,0)$ does not lie in the span of S.

Similarly, we write

$$\begin{aligned} (5,1,-4) &= \alpha(1,1,0) + \beta(0,1,1) \\ &= (\alpha, \alpha + \beta, \beta). \end{aligned}$$

Equating corresponding coordinates gives the system

$$\begin{aligned} \alpha \quad\quad &= 5 \\ \alpha + \beta &= 1 \\ \beta &= -4. \end{aligned}$$

The unique solution is $\alpha = 5$ and $\beta = -4$, which gives

$$(5,1,-4) = 5(1,1,0) - 4(0,1,1),$$

so $(5,1,-4)$ lies in the span of S.

(b) The vectors in $\langle S \rangle$ are those of the form

$$\alpha(1,1,0) + \beta(0,1,1) = (\alpha, \alpha + \beta, \beta),$$

for $\alpha, \beta \in \mathbb{R}$. Thus the span $\langle S \rangle$ consists of all vectors in $\mathbb{R}^3$ whose middle coordinate is the sum of the other two.

Geometrically, $\langle S \rangle$ is the plane $y = x + z$; that is the plane $x - y + z = 0$.

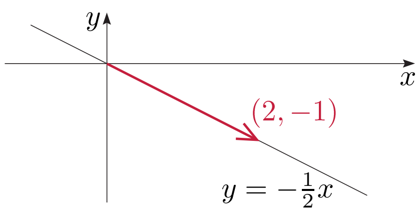
Solution to Additional Exercise C23

(a) We have

$$\begin{aligned} \langle S \rangle &= \{\alpha(2,-1) : \alpha \in \mathbb{R}\} \\ &= \{(2\alpha, -\alpha) : \alpha \in \mathbb{R}\}. \end{aligned}$$

Thus all points of $\langle S \rangle$ lie on the line $y = -\frac{1}{2}x$. All points on the line are of this form and so are in $\langle S \rangle$.

Therefore $\langle S \rangle$ is the line $y = -\frac{1}{2}x$.



(b) We have

$$\begin{aligned}\langle S\rangle &= \{\alpha(0,-1,1)+\beta(0,2,-3) : \alpha,\beta\in\mathbb{R}\}\\ &= \{(0,-\alpha+2\beta,\alpha-3\beta) : \alpha,\beta\in\mathbb{R}\}.\end{aligned}$$

Thus

$$\langle S\rangle \subseteq \{(0,y,z) : y,z\in\mathbb{R}\},$$

so $\langle S\rangle$ is a subset of the (y,z)-plane.

To show that every vector $(0,y,z)$, $y,z\in\mathbb{R}$, belongs to $\langle S\rangle$, we write

$$(0,y,z) = (0,-\alpha+2\beta,\alpha-3\beta).$$

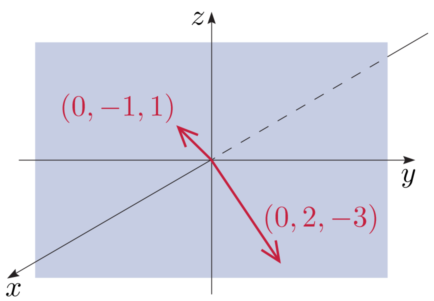
Equating corresponding coordinates, we obtain the system

$$\begin{aligned}-\alpha+2\beta &= y\\ \alpha-3\beta &= z.\end{aligned}$$

The solution is $\beta=-(y+z)$ and $\alpha=-3y-2z$, so

$$(0,y,z) = (-3y-2z)(0,-1,1)-(y+z)(0,2,-3).$$

Hence every point $(0,y,z)$ of the (y,z)-plane belongs to $\langle S\rangle$. Thus $\langle S\rangle$ is the (y,z)-plane.



Solution to Additional Exercise C24

(a) We have

$$\langle S\rangle = \{\alpha(1+2x) : \alpha\in\mathbb{R}\} = \{\alpha+2\alpha x : \alpha\in\mathbb{R}\}$$

A vector $p(x)$ lies in the span of $\{1+2x\}$ if and only if $p(x)=\alpha(1+2x)$. So $\langle S\rangle$ is the set of all vectors of the form $\alpha+2\alpha x$, for some $\alpha\in\mathbb{R}$.

(b) We have

$$\begin{aligned}\langle S\rangle &= \left\{\alpha\begin{pmatrix}1&0\\0&0\end{pmatrix}+\beta\begin{pmatrix}0&0\\0&3\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}\\ &= \left\{\begin{pmatrix}\alpha&0\\0&3\beta\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}.\end{aligned}$$

This is the set of all diagonal matrices in $M_{2,2}$.

(If the question had asked you to *determine* the span here, you could show that any diagonal 2×2 matrix $\mathbf{A}=\begin{pmatrix}a&0\\0&b\end{pmatrix}$ for $a,b\in\mathbb{R}$, belongs to the span since a solution is $\alpha=a$ and $\beta=\frac{1}{3}b$.)

Solution to Additional Exercise C25

Each vector in $\mathbb{R}^2$ can be written as (x,y). To show that (x,y) is in $\{(-1,0),(2,1)\}$, we write

$$\begin{aligned}(x,y) &= \alpha(-1,0)+\beta(2,1)\\ &= (-\alpha,0)+(2\beta,\beta) = (-\alpha+2\beta,\beta).\end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}-\alpha+2\beta &= x\\ \beta &= y,\end{aligned}$$

whose solution is $\alpha=-x+2y$, $\beta=y$.

So any vector (x,y) in $\mathbb{R}^2$ can be written in terms of $(-1,0)$ and $(2,1)$ as

$$(x,y) = (-x+2y)(-1,0)+y(2,1).$$

Therefore $\{(-1,0),(2,1)\}$ spans $\mathbb{R}^2$.

Solution to Additional Exercise C26

(a) This set is not linearly independent, since it contains the zero vector.

(b) We use Strategy C7.

We write

$$\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1) = (0,0,0),$$

which simplifies to

$$(\alpha+\beta,\alpha+\gamma,\beta+\gamma) = (0,0,0).$$

Equating corresponding coordinates gives the system

$$\begin{aligned}\alpha+\beta &= 0\\ \alpha+\gamma &= 0\\ \beta+\gamma &= 0.\end{aligned}$$

From the first equation $\alpha=-\beta$, and substituting this into the second gives $-\beta+\gamma=0$. Adding this to the third equation gives $\gamma=0$, and substitution into the other two equations gives $\alpha=\beta=0$; this is the only solution.

Therefore the set is linearly independent.

(c) Again we use Strategy C7.

We write

$$\alpha(1+2x)+\beta(3x)+\gamma(2-4x)=0+0x,$$

which simplifies to

$$(\alpha+2\gamma)+(2\alpha+3\beta-4\gamma)x=0+0x.$$

Equating coefficients, we obtain the system

$$\begin{aligned} \alpha \quad\quad\;\; + 2\gamma &= 0\\ 2\alpha+3\beta-4\gamma &= 0. \end{aligned}$$

Choosing $\gamma=1$, for example, gives the solution $\alpha=-2$, $\beta=\frac{8}{3}$ and $\gamma=1$, so the set is linearly dependent.

The set $\{1+2x, 3x, 2-4x\}$ is not linearly independent.

(There are other solutions here, for example $\alpha=6$, $\beta=-8$ and $\gamma=-3$.)

Solution to Additional Exercise C27

(Note that this question explicitly asked you to use Strategy C8.)

(a) We check both conditions in Strategy C8.

Neither vector is a multiple of the other, and so these two vectors are linearly independent.

We use Strategy C6.

Each matrix in $M_{2,1}$ can be written as $\begin{pmatrix} x\\ y\end{pmatrix}$, $x,y\in\mathbb{R}$. To show that S spans $M_{2,1}$ we write

$$\begin{pmatrix} x\\ y\end{pmatrix}=\alpha\begin{pmatrix} 1\\ 2\end{pmatrix}+\beta\begin{pmatrix} 1\\ 4\end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha+\;\beta &= x\\ 2\alpha+4\beta &= y. \end{aligned}$$

This system has solution $\alpha=(4x-y)/2$ and $\beta=(y-2x)/2$, so any vector in $M_{2,1}$ can be written as

$$\begin{pmatrix} x\\ y\end{pmatrix}=\frac{4x-y}{2}\begin{pmatrix} 1\\ 2\end{pmatrix}+\frac{y-2x}{2}\begin{pmatrix} 1\\ 4\end{pmatrix}.$$

Therefore this set spans $M_{2,1}$.

Thus this set is a basis for $M_{2,1}$.

(b) We check both conditions in Strategy C8.

Neither vector is a multiple of the other, and so these two vectors are linearly independent.

(If you did not spot this, then you should have found that the only solution to

$$\alpha(2+x)+\beta(3-x^2)=0$$

is $\alpha=\beta=0$ and hence that they are linearly independent.)

We use Strategy C6.

Each polynomial in P_3 can be written as $a+bx+cx^2$ with $a,b,c\in\mathbb{R}$. To show that $a+bx+cx^2$ is in span S we write

$$a+bx+cx^2=\alpha(2+x)+\beta(3-x^2).$$

Equating coefficients, we obtain the system

$$\begin{aligned} 2\alpha+\;3\beta &= a\\ \alpha \quad\quad\; &= b\\ -\beta &= c. \end{aligned}$$

Substituting $\alpha=b$, $\beta=-c$ into the first equation gives $a=2b-3c$. This contradicts the assumption that a, b and c can take any real values, so $\{2+x, 3-x^2\}$ is not a spanning set for P_3.

Therefore the set S is not a basis for P_3.

(c) The second vector is a multiple of the first, so the set $\{1-2i, 2-4i\}$ is linearly dependent.

Therefore the set S is not a basis.

Solution to Additional Exercise C28

(a) The set contains two vectors. They are linearly independent, as neither vector is a multiple of the other. So S is a basis for $\mathbb{R}^2$, by Theorem C25.

(b) Two vectors cannot span $\mathbb{R}^3$, since $\mathbb{R}^3$ has dimension 3. Hence this is not a basis for $\mathbb{R}^3$.

(c) We check both conditions in Strategy C9.

The set S contains 4 vectors.

Using Strategy C7, we write

$$\begin{aligned} &\alpha_1(1,0,0,0)+\alpha_2(1,1,0,0)+\alpha_3(1,1,1,0)\\ &\quad+\alpha_4(1,1,1,1)=(0,0,0,0). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3+\alpha_4 &= 0\\ \alpha_2+\alpha_3+\alpha_4 &= 0\\ \alpha_3+\alpha_4 &= 0\\ \alpha_4 &= 0. \end{aligned}$$

Solving this system gives $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ as the only solution.

Therefore the set is linearly independent.

We have four linearly independent vectors, so S is a basis for $\mathbb{R}^4$ (by Theorem C25).

Solution to Additional Exercise C29

(a) For the basis $E = \{(-3,1),(1,2)\}$, we have

$$\begin{aligned}(2,1)_E &= 2(-3,1)+1(1,2)\\ &= (-6,2)+(1,2)\\ &= (-5,4).\end{aligned}$$

(b) For the basis $E = \{(1,0,2),(-1,1,3),(2,-2,0)\}$, we have

$$\begin{aligned}(1,-2,3)_E &= 1(1,0,2)-2(-1,1,3)+3(2,-2,0)\\ &= (1,0,2)+(2,-2,-6)+(6,-6,0)\\ &= (9,-8,-4).\end{aligned}$$

Solution to Additional Exercise C30

(a) We write

$$(6,5) = \alpha(-3,1)+\beta(1,2).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}-3\alpha + \beta &= 6\\ \alpha + 2\beta &= 5.\end{aligned}$$

Adding three times the second equation to the first gives $7\beta = 21$, so $\beta = 3$. Substituting this into the first equation gives $-3\alpha + 3 = 6$, so $\alpha = -1$.

So

$$\begin{aligned}(6,5) &= -1(-3,1)+3(1,2)\\ &= (-1,3)_E.\end{aligned}$$

(b) We write

$$(3,5,-5) = \alpha(1,2,0)+\beta(-1,3,1)+\gamma(0,2,-2).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha - \beta \phantom{{}+2\gamma} &= 3\\ 2\alpha + 3\beta + 2\gamma &= 5\\ \beta - 2\gamma &= -5.\end{aligned}$$

Adding the second and third equations gives $2\alpha + 4\beta = 0$, and the first gives $\alpha = 3+\beta$, so together we have $6+6\beta = 0$; that is, $\beta = -1$ and hence $\alpha = 2$. Substituting into the third equation gives $\gamma = 2$. So

$$\begin{aligned}(3,5,-5) &= 2(1,2,0)-1(-1,3,1)+2(0,2,-2)\\ &= (2,-1,2)_E.\end{aligned}$$

Solution to Additional Exercise C31

Let the vector space be V and the subset be S, and suppose that conditions (a), (b) and (c) of Theorem C27 hold.

We first notice that the axioms A2, A5, S2, S3, D1 and D2 of V hold for all vectors in V. Since S is a subset of V, each vector in S is also in V, and so these axioms hold for all vectors in S.

We consider the other axioms.

A1 Closure This holds for S, by condition (b).

A3 Additive identity This holds for S because the identity vector $\mathbf{0}$ of V is in S by condition (a), and

$$\mathbf{v}+\mathbf{0} = \mathbf{v} = \mathbf{0}+\mathbf{v}$$

for all vectors $\mathbf{v}$ in V, and, in particular, for all vectors in the subset S.

A4 Additive inverses If $\mathbf{v} \in S$, then $-1\mathbf{v} = -\mathbf{v}$ is also in S, by condition (c). Since S is a subset of V we know that

$$\mathbf{v}+(-\mathbf{v}) = \mathbf{0} = -\mathbf{v}+\mathbf{v},$$

therefore each vector in S also has an inverse in S, so the axiom holds.

S1 Closure This holds, by condition (c).

Since all the axioms hold for S if the conditions of Theorem C27 hold, S is a subspace of V.

Solution to Additional Exercise C32

(a) If $x = y = 0$, then $(x,y,2x+y) = (0,0,0)$, so S contains the zero vector.

Let $\mathbf{v} = (x_1, y_1, 2x_1+y_1)$ and $\mathbf{w} = (x_2, y_2, 2x_2+y_2)$ belong to S. Then

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= (x_1,y_1,2x_1+y_1)+(x_2,y_2,2x_2+y_2)\\ &= (x_1+x_2, y_1+y_2, 2(x_1+x_2)+(y_1+y_2)).\end{aligned}$$

This vector has the correct form for a vector in S, so S is closed under vector addition.

Let $\mathbf{v} = (x,y,2x+y) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(x,y,2x+y)\\ &= (\alpha x, \alpha y, 2(\alpha x)+(\alpha y)).\end{aligned}$$

This vector has the correct form for a vector in S, so S is closed under scalar multiplication.

So S is a subspace of $\mathbb{R}^3$. (It is the plane through the origin with equation $2x+y-z=0$.)

(b) There is no value of x for which $(x, x-3) = (0,0)$, so S does not contain the zero vector.

So S is not a subspace of $\mathbb{R}^2$.

(c) If $x = y = 0$, then

$$(x, y, x+3y, 2x-y) = (0,0,0,0),$$

so S contains the zero vector.

Let $\mathbf{v} = (x_1, y_1, x_1 + 3y_1, 2x_1 - y_1)$ and $\mathbf{w} = (x_2, y_2, x_2 + 3y_2, 2x_2 - y_2)$ belong to S. Then

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (x_1, y_1, x_1 + 3y_1, 2x_1 - y_1) \\ &\quad + (x_2, y_2, x_2 + 3y_2, 2x_2 - y_2) \\ &= (x_1 + x_2, y_1 + y_2, (x_1 + x_2) + 3(y_1 + y_2), \\ &\qquad 2(x_1 + x_2) - (y_1 + y_2)).\end{aligned}$$

This vector has the correct form for a vector in S, so S is closed under vector addition.

Let $\mathbf{v} = (x, y, x+3y, 2x-y) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(x, y, x+3y, 2x-y) \\ &= (\alpha x, \alpha y, (\alpha x) + 3(\alpha y), 2(\alpha x) - (\alpha y)).\end{aligned}$$

This vector has the correct form for a vector in S, so S is closed under scalar multiplication.

So S is a subspace of $\mathbb{R}^4$.

(d) If $a = 0$, then $ax^2 = 0x^2 = \mathbf{0}$, so S contains the zero vector.

Let $\mathbf{v} = a_1x^2$ and $\mathbf{w} = a_2x^2$ belong to S. Then

$$\mathbf{v} + \mathbf{w} = a_1x^2 + a_2x^2 = (a_1 + a_2)x^2.$$

This vector has the correct form for a vector in S, so S is closed under vector addition.

Let $\mathbf{v} = ax^2 \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\mathbf{v} = \alpha(ax^2) = (\alpha a)x^2.$$

This vector has the correct form for a vector in S, so S is closed under scalar multiplication.

So S is a subspace of P_3.

(e) Let $\mathbf{v} = (1,0)$ belong to S and take $\alpha = -1$. Then

$$\alpha\mathbf{v} = -(1,0) = (-1,0),$$

which is not in S.

Therefore S is not closed under scalar multiplication.

So S is not a subspace of $\mathbb{R}^2$.

(f) S is a subspace, by Theorem C28.

Solution to Additional Exercise C33

(a) Any vector in S can be written as

$$(x, y, 2x+y) = x(1,0,2) + y(0,1,1).$$

So $\{(1,0,2), (0,1,1)\}$ spans S. Since this set is linearly independent (the two vectors are not multiples of each other), it is a basis for the subspace, which therefore has dimension 2.

(b) This is not a subspace.

(c) Any vector in S can be written as

$$\begin{aligned}&(x, y, x+3y, 2x-y) \\ &\quad = x(1,0,1,2) + y(0,1,3,-1).\end{aligned}$$

So $\{(1,0,1,2), (0,1,3,-1)\}$ spans S. Since this set is linearly independent (the two vectors are not multiples of each other), it is a basis for the subspace, which therefore has dimension 2.

(d) Any vector in S is of the form ax^2, so it is a multiple of the vector x^2. Therefore $\{x^2\}$ spans S. Since this set is linearly independent, it is a basis for S, which therefore has dimension 1.

(e) This is not a subspace.

(f) We know that

$$\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}\right\}$$

spans S, since S is defined as the span of this set of vectors.

Since these matrices are not multiples of each other, they are linearly independent. So this set is a basis for S, which therefore has dimension 2.

Solution to Additional Exercise C34

(a) $$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= (1\times 2) + (5\times 8) + (-3\times 0) \\ &\quad + (4\times(-7)) + (-7\times(-2)) \\ &= 2 + 40 + 0 - 28 + 14 = 28\end{aligned}$$

(b) $$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_1 &= 1^2 + 5^2 + (-3)^2 + 4^2 + (-7)^2 \\ &= 1 + 25 + 9 + 16 + 49 = 100,\end{aligned}$$

so the magnitude of $\mathbf{v}_1$ is $\sqrt{100} = 10$.

$$\begin{aligned}\mathbf{v}_2 \cdot \mathbf{v}_2 &= 2^2 + 8^2 + 0^2 + (-7)^2 + (-2)^2 \\ &= 4 + 64 + 0 + 49 + 4 = 121,\end{aligned}$$

so the magnitude of $\mathbf{v}_2$ is $\sqrt{121} = 11$.

Solution to Additional Exercise C35

(a) We have

$(5,5,5,5)\cdot(5,-5,-5,5) = 25-25-25+25 = 0,$

$(5,5,5,5)\cdot(5,0,0,-5) = 25+0+0-25 = 0,$

$(5,5,5,5)\cdot(0,5,-5,0) = 0+25-25+0 = 0,$

$(5,-5,-5,5)\cdot(5,0,0,-5) = 25+0+0-25 = 0,$

$(5,-5,-5,5)\cdot(0,5,-5,0) = 0-25+25+0 = 0,$

$(5,0,0,-5)\cdot(0,5,-5,0) = 0+0+0+0 = 0.$

Thus the given vectors form an orthogonal set. Since there are four of them, they form an orthogonal basis for $\mathbb{R}^4$ (by Corollary C33).

(b) Using Strategy C12 (or Theorem C34),

$$\begin{aligned}(5,0,0,0) &= \tfrac{25}{100}(5,5,5,5) + \tfrac{25}{100}(5,-5,-5,5)\\ &\quad + \tfrac{25}{50}(5,0,0,-5) + \tfrac{0}{50}(0,5,-5,0)\\ &= \tfrac{1}{4}(5,5,5,5) + \tfrac{1}{4}(5,-5,-5,5)\\ &\quad + \tfrac{1}{2}(5,0,0,-5) + 0(0,5,-5,0).\end{aligned}$$

(c) $$\begin{aligned}|(5,5,5,5)| &= \sqrt{5^2+5^2+5^2+5^2}\\ &= \sqrt{100} = 10,\end{aligned}$$

$$\begin{aligned}|(5,-5,-5,5)| &= \sqrt{5^2+(-5)^2+(-5)^2+5^2}\\ &= \sqrt{100} = 10,\end{aligned}$$

$$\begin{aligned}|(5,0,0,-5)| &= \sqrt{5^2+0^2+0^2+(-5)^2}\\ &= \sqrt{50} = 5\sqrt{2},\end{aligned}$$

$$\begin{aligned}|(0,5,-5,0)| &= \sqrt{0^2+5^2+(-5)^2+0^2}\\ &= \sqrt{50} = 5\sqrt{2}.\end{aligned}$$

Thus the corresponding orthonormal basis for $\mathbb{R}^4$ is

$$\left\{\frac{1}{2}(1,1,1,1), \frac{1}{2}(1,-1,-1,1), \right.$$
$$\left. \frac{1}{\sqrt{2}}(1,0,0,-1), \frac{1}{\sqrt{2}}(0,1,-1,0)\right\}.$$

Solution to Additional Exercise C36

(a) The plane orthogonal to $(1,-2,2)$ has vector equation $\mathbf{x}\cdot(1,-2,2) = 0$.

Let $\mathbf{w}_1 = (0,1,1)$ and $\mathbf{w}_2 = (2,1,0)$, then

$$\begin{aligned}\mathbf{w}_1\cdot(1,-2,2) &= (0,1,1)\cdot(1,-2,2)\\ &= 0-2+2 = 0,\\ \mathbf{w}_2\cdot(1,-2,2) &= (2,1,0)\cdot(1,-2,2)\\ &= 2-2+0 = 0.\end{aligned}$$

So $\mathbf{w}_1$ and $\mathbf{w}_2$ are both in the plane orthogonal to $(1,-2,2)$. They are not multiples of one another and so are linearly independent.

(b) We set $\mathbf{v}_1 = \mathbf{w}_1 = (0,1,1)$ and

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{v}_1\cdot\mathbf{w}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1\\ &= (2,1,0) - \frac{(0,1,1)\cdot(2,1,0)}{(0,1,1)\cdot(0,1,1)}(0,1,1)\\ &= (2,1,0) - \tfrac{1}{2}(0,1,1)\\ &= \left(2,\tfrac{1}{2},-\tfrac{1}{2}\right)\\ &= \tfrac{1}{2}(4,1,-1).\end{aligned}$$

The required orthogonal basis for the plane is

$$\left\{(0,1,1), \tfrac{1}{2}(4,1,-1)\right\}.$$

(c) An orthogonal basis for $\mathbb{R}^3$ is therefore

$$\left\{(1,-2,2), (0,1,1), \tfrac{1}{2}(4,1,-1)\right\}.$$

(d) We first calculate the magnitude of each basis vector:

$$\begin{aligned}|(1,-2,2)| &= \sqrt{1^2+(-2)^2+2^2}\\ &= \sqrt{9} = 3,\end{aligned}$$

$$\begin{aligned}|(0,1,1)| &= \sqrt{0^2+1^2+1^2}\\ &= \sqrt{2}\end{aligned}$$

and

$$\begin{aligned}\left|\left(2,\tfrac{1}{2},-\tfrac{1}{2}\right)\right| &= \sqrt{2^2+\left(\tfrac{1}{2}\right)^2+\left(-\tfrac{1}{2}\right)^2}\\ &= \sqrt{\tfrac{9}{2}} = \tfrac{3}{2}\sqrt{2}.\end{aligned}$$

Dividing each basis vector by its length, we obtain the orthonormal basis for $\mathbb{R}^3$

$$\left\{\tfrac{1}{3}(1,-2,2), \tfrac{1}{2}\sqrt{2}(0,1,1), \tfrac{1}{3}\sqrt{2}\left(2,\tfrac{1}{2},-\tfrac{1}{2}\right)\right\}.$$

Solution to Additional Exercise C37

Let $\mathbf{w}_1 = (2,2,1,0)$, $\mathbf{w}_2 = (1,2,0,2)$, $\mathbf{w}_3 = (0,1,2,2)$ and $\mathbf{w}_4 = (2,0,2,1)$; then we use the Gram–Schmidt orthogonalisation process.

Let $\mathbf{v}_1 = \mathbf{w}_1 = (2,2,1,0)$,

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1\\
&= (1,2,0,2) - \tfrac{2}{3}(2,2,1,0)\\
&= \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}, 2\right),
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2\\
&= (0,1,2,2) - \tfrac{4}{9}(2,2,1,0)\\
&\quad - \tfrac{2}{3}\left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}, 2\right)\\
&= \left(-\tfrac{2}{3}, -\tfrac{1}{3}, 2, \tfrac{2}{3}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_4}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_4}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2\\
&\quad - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_4}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3\\
&= (2,0,2,1) - \tfrac{2}{3}(2,2,1,0)\\
&\quad - 0\left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}, 2\right) - \tfrac{2}{3}\left(-\tfrac{2}{3}, -\tfrac{1}{3}, 2, \tfrac{2}{3}\right)\\
&= \left(\tfrac{10}{9}, -\tfrac{10}{9}, 0, \tfrac{5}{9}\right).
\end{aligned}$$

Thus an orthogonal basis for $\mathbb{R}^4$ is
$\left\{(2,2,1,0), \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}, 2\right), \left(-\tfrac{2}{3}, -\tfrac{1}{3}, 2, \tfrac{2}{3}\right), \left(\tfrac{10}{9}, -\tfrac{10}{9}, 0, \tfrac{5}{9}\right)\right\}$.

Additional exercises for Unit C3

Section 1

Additional Exercise C38

Determine whether or not each of the following functions is a linear transformation.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (3x + y, 2x - y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x + 1, y)$

Additional Exercise C39

Determine whether or not each of the following functions is a linear transformation.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x, y) \longmapsto (x^2, y, y)$

(b) $t : \mathbb{R}^3 \longmapsto \mathbb{R}^2$
$(x, y, z) \longmapsto (x + y, x - z)$

(c) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$
$(x, y) \longmapsto (x, y, 1, x)$

Additional Exercise C40

Determine whether or not each of the following functions is a linear transformation.

(a) $t : P_3 \longrightarrow P_3$
$p(x) \longmapsto p(x) + p'(x)$

(b) $t : P_3 \longrightarrow P_3$
$p(x) \longmapsto p(x) + 2$

Additional Exercise C41

Let V be the vector space

$$V = \{f(x) : f(x) = ae^x \cos x + be^x \sin x,\ \ a, b \in \mathbb{R}\}.$$

Determine whether or not each of the following functions is a linear transformation.

(a) $t : V \longrightarrow V$
$f(x) \longmapsto f'(x)$

(b) $t : V \longrightarrow \mathbb{R}^2$
$ae^x \cos x + be^x \sin x \longmapsto (2a, b + 1)$

Section 2

Additional Exercise C42

For each of the following linear transformations, find the matrix of t with respect to the standard bases for the domain and codomain.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (3x + y, 2x - y)$

(b) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$(x, y, z) \longmapsto (x + y, x - z)$

Additional Exercise C43

Find the matrix representations of the linear transformation

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + y, 2x + 3y)$$

with respect to the following bases E for the domain and F for the codomain.

(a) $E = \{(1, 1), (0, 1)\}$ $F = \{(1, 0), (0, 1)\}$

(b) $E = \{(1, 0), (0, 1)\}$ $F = \{(2, 1), (1, 3)\}$

(c) $E = \{(1, 2), (1, 1)\}$ $F = \{(1, 0), (3, 1)\}$

Additional Exercise C44

Let

$$V = \{f(x) : f(x) = ae^x \cos x + be^x \sin x,\ \ a, b \in \mathbb{R}\}.$$

Find the matrix of the linear transformation

$$t : V \longrightarrow V$$
$$f(x) \longmapsto f'(x)$$

with respect to the basis $\{e^x \cos x, e^x \sin x\}$ for both the domain and codomain.

Additional Exercise C45

Find the matrix representations of the linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p'(x) \end{aligned}$$

with respect to the following bases E for the domain and F for the codomain.

(a) $E = F = \{1, x, x^2\}$

(b) $E = \{1 + x, 2x, x^2 - x\}$ $\quad F = \{1, x, x^2\}$

(c) $E = \{1, x, x^2\}$ $\quad F = \{1 + x, 2x, x^2 - x\}$

Additional Exercise C46 Challenging

Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix, $\mathbf{B} = (b_{ij})$ be an $n \times p$ matrix and α a scalar. Prove that $(\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$.

Hint: Recall that $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$, where $\mathbf{I}_m$ and $\mathbf{I}_n$ are identity matrices.

Section 3

Additional Exercise C47

Let t and s be the linear transformations

$$\begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x, 2y) \end{aligned} \quad \text{and} \quad \begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + y, 2y). \end{aligned}$$

Determine the following linear transformations.

(a) $s \circ t$ $\quad$ (b) $t \circ s$ $\quad$ (c) $t \circ t$

Additional Exercise C48

Let s and t be the linear transformations

$$\begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x + y, 2x - y) \end{aligned}$$

and

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + y, x - z). \end{aligned}$$

Use the Composition Rule (Theorem C43) and the solution to Additional Exercise C42 to find the matrix representation of the linear transformation $s \circ t$ with respect to the standard bases for the domain and codomain.

Additional Exercise C49

Let

$V = \{f(x) : f(x) = ae^x \cos x + be^x \sin x,\ a, b \in \mathbb{R}\}$

and let t be the linear transformation

$$\begin{aligned} t : V &\longrightarrow V \\ f(x) &\longmapsto f'(x). \end{aligned}$$

Use the Composition Rule (Theorem C43) to find the matrix representation of the linear transformation

$$\begin{aligned} t \circ t : V &\longrightarrow V \\ f(x) &\longmapsto f''(x) \end{aligned}$$

with respect to the basis $E = \{e^x \cos x, e^x \sin x\}$ for the domain and codomain.

Hence find the image of a function

$$f(x) = ae^x \cos x + be^x \sin x$$

under $t \circ t$.

You found in Additional Exercise C44 that the matrix of t with respect to these bases is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Additional Exercise C50

Determine whether or not each of the following linear transformations is invertible. Find the inverse of each invertible linear transformation.

(a) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x + y, y) \end{aligned}$

(b) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x + 6y, x + 3y) \end{aligned}$

(c) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^4 \\ (x, y) &\longmapsto (x, y, x, y) \end{aligned}$

(d) $\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + z, 2x + y + 3z, 2y + z) \end{aligned}$

Additional Exercise C51

Determine whether or not the following linear transformation is invertible.

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p'(x) \end{aligned}$$

(You found in Additional Exercise C45(a) that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} a+b \\ b+2c \\ c \end{pmatrix}_F$$

is the matrix representation of t with respect to the standard basis $\{1, x, x^2\}$ for both the domain and codomain.)

Additional Exercise C52

Let

$$V = \{f(x) : f(x) = ae^x \cos x + be^x \sin x,\ \ a, b \in \mathbb{R}\}.$$

Write down an isomorphism:

(a) from V to $\mathbb{R}^2$ (b) from V to P_2.

Additional Exercise C53 Challenging

Prove that the set of invertible linear transformations from $\mathbb{R}^n$ to itself forms a group under composition.

Section 4

Additional Exercise C54

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + y, x - z). \end{aligned}$$

(a) What can you deduce from the Dimension Theorem (Theorem C53) about whether t is one-to-one and/or onto?

(b) Find the kernel of t and the dimension of the kernel.

(c) Hence find $\operatorname{Im} t$.

(d) Is t one-to-one and/or onto?

Additional Exercise C55

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, y + z, x + z). \end{aligned}$$

(a) Find $\operatorname{Im} t$.

(b) Find $\operatorname{Ker} t$.

(c) Use your answers to parts (a) and (b) to determine the number of solutions to the following system of linear equations.

$$\begin{aligned} x + 2y + 3z &= 4 \\ y + z &= 1 \\ x + z &= 2 \end{aligned}$$

Additional Exercise C56

Let

$$V = \{f(x) : f(x) = ae^x \cos x + be^x \sin x,\ \ a, b \in \mathbb{R}\},$$

and let t be the linear transformation

$$\begin{aligned} t : V &\longrightarrow V \\ f(x) &\longmapsto f'(x). \end{aligned}$$

(a) Find a basis for $\operatorname{Im} t$, and state the dimension of $\operatorname{Im} t$.

(b) Hence find $\operatorname{Ker} t$.

(c) Is t one-to-one and/or onto?

Solutions to additional exercises for Unit C3

Solution to Additional Exercise C38

We use Strategy C14.

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} &t(\mathbf{v}_1 + \mathbf{v}_2) \\ &= t(x_1 + x_2, y_1 + y_2) \\ &= (3(x_1 + x_2) + y_1 + y_2, 2(x_1 + x_2) - (y_1 + y_2)) \end{aligned}$$

and

$$\begin{aligned} &t(\mathbf{v}_1) + t(\mathbf{v}_2) \\ &= (3x_1 + y_1, 2x_1 - y_1) + (3x_2 + y_2, 2x_2 - y_2) \\ &= (3(x_1 + x_2) + y_1 + y_2, 2(x_1 + x_2) - (y_1 + y_2)). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y)$ be a vector in $\mathbb{R}^2$, and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} t(\alpha\mathbf{v}) &= t(\alpha x, \alpha y) \\ &= (3\alpha x + \alpha y, 2\alpha x - \alpha y) \end{aligned}$$

and

$$\begin{aligned} \alpha\, t(\mathbf{v}) &= \alpha(3x + y, 2x - y) \\ &= (3\alpha x + \alpha y, 2\alpha x - \alpha y). \end{aligned}$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

(b) Since $t(\mathbf{0}) = t(0,0) = (1,0) \neq \mathbf{0}$, it follows that t is not a linear transformation.

Solution to Additional Exercise C39

We use Strategy C14.

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2) \\ &= ((x_1 + x_2)^2, y_1 + y_2, y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= (x_1^2, y_1, y_1) + (x_2^2, y_2, y_2) \\ &= (x_1^2 + x_2^2, y_1 + y_2, y_1 + y_2). \end{aligned}$$

These expressions are not equal in general, so LT1 is not satisfied.

Thus t is not a linear transformation.

(b) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3.$$

In $\mathbb{R}^3$, let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$. Then

$$\begin{aligned} &t(\mathbf{v}_1 + \mathbf{v}_2) \\ &= t(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - z_1 - z_2) \end{aligned}$$

and

$$\begin{aligned} &t(\mathbf{v}_1) + t(\mathbf{v}_2) \\ &= (x_1 + y_1, x_1 - z_1) + (x_2 + y_2, x_2 - z_2) \\ &= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - z_1 - z_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^3,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y, z)$ be a vector in $\mathbb{R}^3$, and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} t(\alpha\mathbf{v}) &= t(\alpha x, \alpha y, \alpha z) \\ &= (\alpha x + \alpha y, \alpha x - \alpha z) \end{aligned}$$

and

$$\begin{aligned} \alpha\, t(\mathbf{v}) &= \alpha(x + y, x - z) \\ &= (\alpha x + \alpha y, \alpha x - \alpha z). \end{aligned}$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

(c) Since $t(\mathbf{0}) = t(0,0) = (0,0,1,0) \neq \mathbf{0}$, it follows that t is not a linear transformation.

Solution to Additional Exercise C40

We use Strategy C14.

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(p(x) + q(x)) = t(p(x)) + t(q(x)), \quad \text{for all } p(x), q(x) \in P_3.$$

Let $p(x), q(x) \in P_3$. Then

$$t(p(x) + q(x)) = p(x) + q(x) + p'(x) + q'(x)$$

and

$$\begin{aligned} t(p(x)) + t(q(x)) &= p(x) + p'(x) + q(x) + q'(x) \\ &= p(x) + q(x) + p'(x) + q'(x). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha p(x)) = \alpha\, t(p(x)), \quad \text{for all } p(x) \in P_3,\ \alpha \in \mathbb{R}.$$

Let $p(x) \in P_3$ and $\alpha \in \mathbb{R}$. Then

$$t(\alpha p(x)) = \alpha p(x) + \alpha p'(x)$$

and

$$\begin{aligned} \alpha\, t(p(x)) &= \alpha(p(x) + p'(x)) \\ &= \alpha p(x) + \alpha p'(x). \end{aligned}$$

These expressions are equal, so LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t is a linear transformation.

(b) The zero element of P_3 is $p(x) = 0$ which maps to the polynomial $q(x) = 2$. Thus $t(\mathbf{0}) \neq \mathbf{0}$, so it follows that t is not a linear transformation.

Solution to Additional Exercise C41

We use Strategy C14.

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(f(x) + g(x)) = t(f(x)) + t(g(x)), \quad \text{for all } f(x), g(x) \in V.$$

Let $f(x), g(x) \in V$. Then

$$t(f(x) + g(x)) = f'(x) + g'(x)$$

and

$$t(f(x)) + t(g(x)) = f'(x) + g'(x).$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha f(x)) = \alpha\, t(f(x)), \quad \text{for all } f(x) \in V,\ \alpha \in \mathbb{R}.$$

Let $f(x) \in V$ and $\alpha \in \mathbb{R}$. Then

$$t(\alpha f(x)) = \alpha f'(x)$$

and

$$\alpha\, t(f(x)) = \alpha f'(x).$$

These expressions are equal, so LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t is a linear transformation.

(b) Since

$$t(\mathbf{0}) = t(0e^x \cos x + 0e^x \sin x) = (0, 1) \neq \mathbf{0},$$

it follows that t is not a linear transformation.

Solution to Additional Exercise C42

We follow Strategy C15.

(a) We find the images of the vectors in the domain basis $E = \{(1,0),(0,1)\}$:

$$t(1,0) = (3,2), \quad t(0,1) = (1,-1).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,0) = (3,2)_F, \quad t(0,1) = (1,-1)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}.$$

(b) We find the images of the vectors in the domain basis $E = \{(1,0,0),(0,1,0),(0,0,1)\}$:

$$t(1,0,0) = (1,1), \quad t(0,1,0) = (1,0),$$
$$t(0,0,1) = (0,-1).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,0,0) = (1,1)_F, \quad t(0,1,0) = (1,0)_F,$$
$$t(0,0,1) = (0,-1)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Solution to Additional Exercise C43

We follow Strategy C15.

(a) We find the images of the vectors in the domain basis $E = \{(1,1),(0,1)\}$:

$$t(1,1) = (2,5), \quad t(0,1) = (1,3).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,1) = (2,5)_F, \quad t(0,1) = (1,3)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

Thus the matrix representation of t with respect to the non-standard domain basis E and the standard codomain basis F is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_E \longmapsto \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_E = \begin{pmatrix} 2v_1 + v_2 \\ 5v_1 + 3v_2 \end{pmatrix}_F.$$

(b) We find the images of the vectors in the domain basis $E = \{(1,0),(0,1)\}$:

$$t(1,0) = (1,2), \quad t(0,1) = (1,3).$$

We find the F-coordinates of each of these image vectors, where $F = \{(2,1),(1,3)\}$.

For the first image vector we need $a, b \in \mathbb{R}$ such that

$$(1,2) = (a,b)_F.$$

Since

$$(a,b)_F = a(2,1) + b(1,3) = (2a + b, a + 3b),$$

by equating coordinates we obtain the system

$$\begin{aligned} 2a + b &= 1 \\ a + 3b &= 2. \end{aligned}$$

Solving, we have $a = \frac{1}{5}$ and $b = \frac{3}{5}$, so $(1,2) = \left(\frac{1}{5}, \frac{3}{5}\right)_F$. Therefore

$$t(1,0) = \left(\tfrac{1}{5}, \tfrac{3}{5}\right)_F.$$

For the second image vector we have

$$t(0,1) = (1,3) = (0,1)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{3}{5} & 1 \end{pmatrix}.$$

Thus the matrix representation of t with respect to the standard domain basis E and the non-standard codomain basis F is

$$\begin{pmatrix} x \\ y \end{pmatrix}_E \longmapsto \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{3}{5} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_E = \begin{pmatrix} \frac{1}{5}x \\ \frac{3}{5}x + y \end{pmatrix}_F.$$

(c) We find the images of the vectors in the domain basis $E = \{(1,2),(1,1)\}$:

$$t(1,2) = (3,8), \quad t(1,1) = (2,5).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(3,1)\}$.

For the first image vector we need $a, b \in \mathbb{R}$ such that

$$(3,8) = (a,b)_F.$$

Since

$$(a,b)_F = a(1,0) + b(3,1) = (a + 3b, b),$$

by equating coordinates we obtain the system

$$\begin{aligned} a + 3b &= 3 \\ b &= 8. \end{aligned}$$

Solving, we have $a = -21$ and $b = 8$, so $(3,8) = (-21,8)_F$. Therefore

$$t(1,2) = (-21,8)_F.$$

For the second image vector we need $c, d \in \mathbb{R}$ such that

$$(2,5) = (c,d)_F.$$

Since

$$(c,d)_F = c(1,0) + d(3,1) = (c + 3d, d),$$

by equating coordinates we obtain the system

$$\begin{aligned} c + 3d &= 2 \\ d &= 5. \end{aligned}$$

Solving, we have $c = -13$ and $d = 5$, so $(2,5) = (-13,5)_F$. Therefore

$$t(1,1) = (-13,5)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} -21 & -13 \\ 8 & 5 \end{pmatrix}.$$

Hence the matrix representation of t with respect to the non-standard bases E and F is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_E \longmapsto \begin{pmatrix} -21 & -13 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_E = \begin{pmatrix} -21v_1 - 13v_2 \\ 8v_1 + 5v_2 \end{pmatrix}_F.$$

Solution to Additional Exercise C44

We follow Strategy C15.

We find the images of the vectors in the domain basis $E = \{e^x \cos x, e^x \sin x\}$:

$$\begin{aligned} t(e^x \cos x) &= e^x \cos x - e^x \sin x,\\ t(e^x \sin x) &= e^x \sin x + e^x \cos x. \end{aligned}$$

We find the F-coordinates of each of these image vectors, where $F = \{e^x \cos x, e^x \sin x\}$:

$$t(e^x \cos x) = (1,-1)_F, \quad t(e^x \sin x) = (1,1)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution to Additional Exercise C45

We follow Strategy C15.

(a) We find the images of the vectors in the domain basis $E = \{1, x, x^2\}$:

$$\begin{aligned} &t(1) = 1+0 = 1, \quad t(x) = x+1,\\ &t(x^2) = x^2 + 2x. \end{aligned}$$

We find the F-coordinates of each of these image vectors, where $F = \{1, x, x^2\}$:

$$\begin{aligned} &t(1) = (1,0,0)_F, \quad t(x) = (1,1,0)_F,\\ &t(x^2) = (0,2,1)_F. \end{aligned}$$

So the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ b+2c \\ c \end{pmatrix}.$$

(b) We find the images of the vectors in the domain basis $E = \{1+x, 2x, x^2 - x\}$:

$$\begin{aligned} &t(1+x) = 1+x+1 = x+2,\\ &t(2x) = 2x+2,\\ &t(x^2 - x) = x^2 - x + 2x - 1 = x^2 + x - 1. \end{aligned}$$

We find the F-coordinates of each of these image vectors, where $F = \{1, x, x^2\}$:

$$\begin{aligned} 2+x &= (2,1,0)_F,\\ 2+2x &= (2,2,0)_F,\\ -1+x+x^2 &= (-1,1,1)_F. \end{aligned}$$

Hence the matrix of t with respect to the non-standard basis E and standard basis F is

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix representation of t with respect to the bases E and F is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 2 & 2 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} 2a+2b-c \\ a+2b+c \\ c \end{pmatrix}_F.$$

(c) We find the images of the vectors in the domain basis $E = \{1, x, x^2\}$:

$$\begin{aligned} &t(1) = 1+0 = 1, \quad t(x) = x+1,\\ &t(x^2) = x^2 + 2x. \end{aligned}$$

We find the F-coordinates of each of these image vectors, where $F = \{1+x, 2x, x^2 - x\}$.

For the first image vector we need $a, b, c \in \mathbb{R}$ such that

$$1 = (a,b,c)_F.$$

Since

$$\begin{aligned} (a,b,c)_F &= a(1+x) + b(2x) + c(x^2 - x)\\ &= a + (a+2b-c)x + cx^2, \end{aligned}$$

equating coefficients we obtain the system

$$\begin{aligned} a \phantom{{}+2b-c} &= 1\\ a + 2b - c &= 0\\ c &= 0. \end{aligned}$$

Solving, we have $a = 1$, $b = -\frac{1}{2}$ and $c = 0$, so $1 = \left(1, -\frac{1}{2}, 0\right)_F$. Therefore

$$t(1) = \left(1, -\tfrac{1}{2}, 0\right)_F.$$

Similarly,

$$t(x) = (1,0,0)_F, \quad t(x^2) = \left(0, \tfrac{3}{2}, 1\right)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix representation of t with respect to the standard basis E and the non-standard basis F is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} a+b \\ -\frac{1}{2}a + \frac{3}{2}c \\ c \end{pmatrix}_F .$$

Solution to Additional Exercise C46

Using the identity matrix $\mathbf{I}_m$, we have

$$\begin{aligned}(\alpha \mathbf{I}_m)\mathbf{A} &= \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix} \\ &= \alpha \mathbf{A}.\end{aligned}$$

Using the identity matrix $\mathbf{I}_n$,

$$\begin{aligned}\mathbf{A}(\alpha \mathbf{I}_n) &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}.\end{aligned}$$

Therefore $(\alpha \mathbf{I}_m)\mathbf{A} = \mathbf{A}(\alpha \mathbf{I}_n)$, and hence

$$\begin{aligned}(\alpha \mathbf{A})\mathbf{B} &= (\alpha \mathbf{I}_m)\mathbf{A}\mathbf{B} \\ &= \mathbf{A}(\alpha \mathbf{I}_n)\mathbf{B} \\ &= \mathbf{A}(\alpha \mathbf{B}),\end{aligned}$$

as required.

Solution to Additional Exercise C47

(a) We have

$$\begin{aligned}s(t(x,y)) &= s(x+y, 2y) \\ &= (3(x+y), 2(2y)) \\ &= (3x+3y, 4y).\end{aligned}$$

Thus

$$\begin{aligned} s \circ t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (3x+3y, 4y).\end{aligned}$$

(b) We have

$$\begin{aligned}t(s(x,y)) &= t(3x, 2y) \\ &= (3x+2y, 2(2y)) \\ &= (3x+2y, 4y).\end{aligned}$$

Thus

$$\begin{aligned} t \circ s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (3x+2y, 4y).\end{aligned}$$

(c) We have

$$\begin{aligned}t(t(x,y)) &= t(x+y, 2y) \\ &= ((x+y)+2y, 2(2y)) \\ &= (x+3y, 4y).\end{aligned}$$

Thus

$$\begin{aligned} t \circ t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+3y, 4y).\end{aligned}$$

Solution to Additional Exercise C48

It follows from the Composition Rule (Theorem C43) that the matrix of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Thus the matrix representation of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{aligned} s \circ t &: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto \begin{pmatrix} 4 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x+3y-z \\ x+2y+z \end{pmatrix}.\end{aligned}$$

Solution to Additional Exercise C49

It follows from the Composition Rule (Theorem C43) that the matrix of $t \circ t$ with respect to the basis $E = \{e^x \cos x, e^x \sin x\}$ for both the domain and codomain is

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Thus the matrix representation of $t \circ t$ with respect to the basis $E = \{e^x \cos x, e^x \sin x\}$ for both the domain and codomain is

$$\begin{aligned} t \circ t &: V \longrightarrow V \\ \begin{pmatrix} a \\ b \end{pmatrix}_E &\longmapsto \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_E = \begin{pmatrix} 2b \\ -2a \end{pmatrix}_E .\end{aligned}$$

So the function $f(x) = ae^x \cos x + be^x \sin x$ in V maps to the function $g(x) = 2be^x \cos x - 2ae^x \sin x$ in V under the linear transformation $t \circ t$.

(We could find this directly by differentiating $ae^x \cos x + be^x \sin x$ twice. The matrix multiplication is, however, very simple and is possibly easier than differentiating. To obtain higher derivatives, we could repeatedly apply the Composition Rule.)

Solution to Additional Exercise C50

(a) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0) = (3,0), \quad t(0,1) = (1,1).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ y \end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have

$$\det \mathbf{A} = \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3.$$

Since $\det \mathbf{A} \neq 0$, t is invertible.

We now find the inverse function of t, $t^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. According to Strategy C16, t^{-1} has the matrix representation $\mathbf{v} \longmapsto \mathbf{A}^{-1}\mathbf{v}$, with respect to the standard bases for the domain and codomain. Since

$$\mathbf{A}^{-1} = \tfrac{1}{3} \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 \end{pmatrix},$$

it follows that t^{-1} has the matrix representation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x - \frac{1}{3}y \\ y \end{pmatrix}.$$

So t^{-1} is the linear transformation

$$\begin{aligned} t^{-1} &: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto \left(\tfrac{1}{3}x - \tfrac{1}{3}y, y\right). \end{aligned}$$

(b) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0) = (2,1), \quad t(0,1) = (6,3).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 6y \\ x + 3y \end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}.$$

We have

$$\det \mathbf{A} = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 6 - 6 = 0.$$

Since $\det \mathbf{A} = 0$, t is not invertible.

(c) Since t is a linear transformation between two vector spaces of different dimensions, it follows from Corollary C46 that t is not invertible.

(d) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0,0) = (1,2,0), \quad t(0,1,0) = (0,1,2),$$
$$t(0,0,1) = (1,3,1).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + z \\ 2x + y + 3z \\ 2y + z \end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \det \mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 0 + 1 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 1(1-6) + 1(4-0) \\ &= -5 + 4 = -1. \end{aligned}$$

Since $\det \mathbf{A} \neq 0$, t is invertible.

We now find the inverse function of t.

According to Strategy C16, t^{-1} has the matrix representation $\mathbf{v} \longmapsto \mathbf{A}^{-1}\mathbf{v}$, with respect to the standard bases for the domain and codomain.

Using row-reduction, we find that

$$\mathbf{A}^{-1} = \begin{pmatrix} 5 & -2 & 1 \\ 2 & -1 & 1 \\ -4 & 2 & -1 \end{pmatrix},$$

so t^{-1} has the matrix representation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 5 & -2 & 1 \\ 2 & -1 & 1 \\ -4 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x - 2y + z \\ 2x - y + z \\ -4x + 2y - z \end{pmatrix}.$$

So t^{-1} is the linear transformation

$$\begin{aligned} t^{-1} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (5x - 2y + z, 2x - y + z, \\ &\qquad -4x + 2y - z). \end{aligned}$$

Solution to Additional Exercise C51

Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

We evaluate the determinant of the matrix of t:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} + 0 \\ &= 1(1 - 0) - 1(0 - 0) + 0 \\ &= 1. \end{aligned}$$

Since $\det \mathbf{A} \neq 0$, t is invertible.

Solution to Additional Exercise C52

(a) An isomorphism is

$$\begin{aligned} t : V &\longrightarrow \mathbb{R}^2 \\ ae^x \cos x + be^x \sin x &\longmapsto (a, b). \end{aligned}$$

(The matrix of t with respect to the basis $E = \{e^x \cos x, e^x \sin x\}$ for V and the standard basis for $\mathbb{R}^2$ is $\mathbf{I}_2$. Since t has a matrix representation, it is a linear transformation, by Theorem C41. Since $\mathbf{I}_2$ is invertible, it follows from the Inverse Rule that t is an invertible linear transformation; that is, t is an isomorphism.)

(b) An isomorphism is

$$\begin{aligned} t : V &\longrightarrow P_2 \\ ae^x \cos x + be^x \sin x &\longmapsto a + bx. \end{aligned}$$

(The matrix of t with respect to the basis $E = \{e^x \cos x, e^x \sin x\}$ for V and the standard basis for P_2 is $\mathbf{I}_2$.)

Solution to Additional Exercise C53

We show that, under the operation of composition, the set G of invertible linear transformations from $\mathbb{R}^n$ to itself satisfies the four group axioms.

G1 Closure Let $s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be two invertible linear transformations in G. Let $\mathbf{A}$ be the matrix of t and let $\mathbf{B}$ be the matrix of s, with respect to the standard bases for the domain and codomain. Then it follows from the Composition Rule that $\mathbf{BA}$ is the matrix of the linear transformation $s \circ t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with respect to the standard bases for the domain and codomain.

Since t and s are invertible, it follows from the Inverse Rule that $\mathbf{A}$ and $\mathbf{B}$ are invertible. Thus $\mathbf{BA}$ is invertible, since $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$. It follows from the Inverse Rule that $s \circ t$ is invertible. Thus $s \circ t$ belongs to G; that is, G is closed under composition.

G2 Associativity We know that composition of functions is associative.

G3 Identity We claim that the identity transformation

$$\begin{aligned} i : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{v} &\longmapsto \mathbf{v} \end{aligned}$$

is the identity element of G. The matrix of i is $\mathbf{I}_n$, with respect to the standard bases for the domain and codomain.

Since $\mathbf{I}_n$ is invertible, it follows from the Inverse Rule that i is invertible; that is, i belongs to G.

Suppose that $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to G and $\mathbf{A}$ is the matrix of t with respect to the standard bases for the domain and codomain. Since $\mathbf{AI}_n = \mathbf{A} = \mathbf{I}_n\mathbf{A}$, it follows from the Composition Rule that $t \circ i = t = i \circ t$.

So the identity transformation is the identity element of G.

G4 Inverses Suppose that $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to G. It follows from the Inverse Rule that the inverse function of t, $t^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, is also a linear transformation. It is invertible, since the inverse function of t^{-1} is just t. So the inverse of t belongs to G.

We have shown that, under the operation of composition, G satisfies the four group axioms and is therefore a group.

Solution to Additional Exercise C54

(a) Since the dimension of the codomain is less than the dimension of the domain, the Dimension Theorem tells us that $\dim(\operatorname{Ker} t) \geq 1$; that is, $\operatorname{Ker} t \neq \{\mathbf{0}\}$. Therefore, t is not one-to-one by Proposition C52. (The Dimension Theorem tells us nothing about whether t is onto or not.)

(b) The kernel of t is the set of vectors (x, y, z) in $\mathbb{R}^3$ that satisfy

$$t(x, y, z) = \mathbf{0},$$

that is,

$$(x + y, x - z) = (0, 0).$$

Equating coordinates, we obtain the system

$$\begin{aligned} x + y \phantom{{}-z} &= 0 \\ x \phantom{{}+y} - z &= 0. \end{aligned}$$

Assigning the parameter k to the unknown z, we obtain

$$x = k, \quad y = -k, \quad z = k.$$

So the kernel of t is

$$\operatorname{Ker} t = \{(k, -k, k) : k \in \mathbb{R}\},$$

that is, $\operatorname{Ker} t$ is the line through $(0, 0, 0)$ and $(1, -1, 1)$. Thus

$$\dim(\operatorname{Ker} t) = 1.$$

(c) Since the dimension of the domain of t is 3, it follows from the Dimension Theorem that

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 3.$$

Since $\dim(\operatorname{Ker} t) = 1$, it follows that

$$\dim(\operatorname{Im} t) = 2.$$

Thus $\operatorname{Im} t$ is a two-dimensional subspace of the codomain $\mathbb{R}^2$ and is hence equal to $\mathbb{R}^2$.

(d) We saw in part (a) that t is not one-to-one; this was confirmed in part (b).

We saw in part (c) that $\operatorname{Im} t$ is the whole of the codomain $\mathbb{R}^2$; that is, t is onto.

Solution to Additional Exercise C55

(a) We follow Strategy C17.

We take the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for the domain $\mathbb{R}^3$.

We determine the images of these basis vectors:

$$t(1, 0, 0) = (1, 0, 1), \quad t(0, 1, 0) = (2, 1, 0),$$
$$t(0, 0, 1) = (3, 1, 1).$$

The set $\{(1, 0, 1), (2, 1, 0), (3, 1, 1)\}$ is not linearly independent. In fact,

$$(3, 1, 1) = (1, 0, 1) + (2, 1, 0),$$

so we discard $(3, 1, 1)$ to give the set $\{(1, 0, 1), (2, 1, 0)\}$.

The vectors $(1, 0, 1)$ and $(2, 1, 0)$ are linearly independent, so $\{(1, 0, 1), (2, 1, 0)\}$ is a basis for $\operatorname{Im} t$.

Thus $\operatorname{Im} t$ is a two-dimensional subspace of the codomain $\mathbb{R}^3$; that is, $\operatorname{Im} t$ is a plane through the origin with equation

$$ax + by + cz = 0,$$

for some $a, b, c \in \mathbb{R}$. Since the basis vectors $(1, 0, 1)$ and $(2, 1, 0)$ belong to $\operatorname{Im} t$, the values a, b and c satisfy the system

$$\begin{aligned} a \phantom{{}+b} + c &= 0 \\ 2a + b \phantom{{}+c} &= 0. \end{aligned}$$

Solving, we have $c = -a$ and $b = -2a$. So $\operatorname{Im} t$ is the plane with equation

$$ax - 2ay - az = 0$$

or, equivalently,

$$x - 2y - z = 0.$$

(b) The kernel of t is the set of vectors (x, y, z) in $\mathbb{R}^3$ that satisfy

$$t(x, y, z) = \mathbf{0},$$

that is,

$$(x + 2y + 3z, y + z, x + z) = (0, 0, 0).$$

Equating coordinates we obtain

$$\begin{aligned} x + 2y + 3z &= 0 \\ y + z &= 0 \\ x \phantom{{}+2y} + z &= 0. \end{aligned}$$

The last two equations give $x = -z$ and $y = -z$. The first equation is also satisfied for these values of x, y and z.

Assigning the parameter k to the unknown z, we obtain

$$x = -k, \quad y = -k, \quad z = k.$$

So the kernel of t is

$$\operatorname{Ker} t = \{(-k, -k, k) : k \in \mathbb{R}\},$$

that is, $\operatorname{Ker} t$ is the line through $(0, 0, 0)$ and $(-1, -1, 1)$.

(c) We see that x, y and z satisfy this system of linear equations precisely when

$$t(x, y, z) = (4, 1, 2).$$

Since

$$4 - 2(1) - 2 = 0,$$

it follows from part (a) that $(4, 1, 2)$ belongs to $\operatorname{Im} t$. So the system of linear equations has at least one solution.

We know from part (b) that $\operatorname{Ker} t \neq \{\mathbf{0}\}$; thus the system of linear equations has infinitely many solutions.

Solution to Additional Exercise C56

(a) We follow Strategy C17.

We take the basis $E = \{e^x \cos x, e^x \sin x\}$ for the domain V.

We determine the images of these basis vectors:

$$t(e^x \cos x) = e^x \cos x - e^x \sin x,$$
$$t(e^x \sin x) = e^x \sin x + e^x \cos x.$$

The set $\{e^x \cos x - e^x \sin x, e^x \sin x + e^x \cos x\}$ is linearly independent, so it is a basis for $\operatorname{Im} t$.

The basis has two elements, so $\dim(\operatorname{Im} t) = 2$.

Thus $\operatorname{Im} t$ is a two-dimensional subspace of the codomain V.

(b) Since the dimension of the domain V is 2, it follows from the Dimension Theorem that

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2.$$

We know from part (a) that $\dim(\operatorname{Im} t) = 2$. It follows that $\dim(\operatorname{Ker} t) = 0$, that is,

$$\operatorname{Ker} t = \{\mathbf{0}\}.$$

(c) Since $\operatorname{Ker} t = \{\mathbf{0}\}$, it follows that t is one-to-one (from Proposition C52).

Since V is two-dimensional, it follows that $\operatorname{Im} t$ is the whole of V and t is onto (from Proposition C49).

Additional exercises for Unit C4

Section 1

Additional Exercise C57

Let $\mathbf{v}$ be an eigenvector of a linear transformation t with corresponding eigenvalue λ. Prove that if $k \neq 0$, then $k\mathbf{v}$ is also an eigenvector of t with corresponding eigenvalue λ.

Additional Exercise C58

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$t(x, y) = (-2x + 7y, x + 4y).$$

Find the eigenvalues and eigenvectors of t.

Additional Exercise C59

Let $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x, y, z) = (3x + 2y + 2z, -2x - 2y - 2z, x + 2y + 2z).$$

Find the eigenvalues and eigenvectors of t.

Additional Exercise C60

Prove that the sum of the eigenvalues of a 2×2 square matrix is equal to the sum of the diagonal entries (that is, Proposition C56 for 2×2 matrices) as follows.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Find the characteristic equation of $\mathbf{A}$.

Show that the sum of the eigenvalues is equal to $a + d$; that is, the sum of the diagonal entries of $\mathbf{A}$.

Additional Exercise C61

Let $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x, y, z) = (-2z, x + 2y + z, x + 3z).$$

Find all the eigenspaces of t. For each one, specify a basis and state the dimension of the eigenspace.

Section 2

Additional Exercise C62

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$t(x, y) = (x + y, x + y).$$

(a) Write down the matrix $\mathbf{A}$ of t with respect to the standard basis for $\mathbb{R}^2$.

(b) Find an eigenvector basis E of t.

(c) Find the matrix $\mathbf{D}$ of t with respect to the basis E.

(d) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

Additional Exercise C63

Use the solution to Additional Exercise C58 to diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 7 \\ 1 & 4 \end{pmatrix}.$$

Additional Exercise C64

Use the solution to Additional Exercise C59 to diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}.$$

Additional Exercise C65

Use the solution to Additional Exercise C61 to diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Section 3

Additional Exercise C66

Orthogonally diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}.$$

Additional Exercise C67

Orthogonally diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Additional Exercise C68

Orthogonally diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}.$$

Additional Exercise C69 Challenging

Prove that if a matrix $\mathbf{A}$ is orthogonally diagonalisable, then $\mathbf{A}$ is symmetric.

Additional Exercise C70

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \end{pmatrix}.$$

(a) Show that $\mathbf{A}$ is orthogonal.

(b) Write down $\mathbf{A}^{-1}$.

(c) Show that $\mathbf{A}$ represents a rotation of $\mathbb{R}^3$.

Section 4

Additional Exercise C71

Write the non-degenerate conic with equation

$$x^2 - 2y^2 - 4x - 12y - 18 = 0$$

in standard form. Is this conic an ellipse, a parabola or a hyperbola?

Additional Exercise C72

Write the non-degenerate conic with equation

$$5x^2 - 2xy + 5y^2 - 1 = 0$$

in standard form. Is this conic an ellipse, a parabola or a hyperbola?

(You orthogonally diagonalised the matrix $\mathbf{A}$ of this conic in Additional Exercise C66.)

Additional Exercise C73

Write the quadric with equation

$$2xy - 6x + 10y + z - 30 = 0$$

in standard form. Which of the six types of quadric does this represent?

(You orthogonally diagonalised the matrix $\mathbf{A}$ of this quadric in Additional Exercise C67.)

Additional Exercise C74

Write the quadric with equation

$$x^2 + y^2 + x - z = 0$$

in standard form. Which of the six types of quadric does this represent?

Additional Exercise C75

Write the quadric with equation

$$2xy + z = 0$$

in standard form. Which of the six types of quadric does this represent?

(Use your answer from Additional Exercise C73.)

Solutions to additional exercises for Unit C4

Solution to Additional Exercise C57

Since $\mathbf{v}$ is an eigenvector of the linear transformation t with corresponding eigenvalue λ, we have

$$t(\mathbf{v}) = \lambda\mathbf{v}.$$

Let k be any non-zero number. Then

$$t(k\mathbf{v}) = k\,t(\mathbf{v}) = k(\lambda\mathbf{v}) = \lambda(k\mathbf{v}),$$

so $k\mathbf{v}$ is also an eigenvector of the linear transformation t with corresponding eigenvalue λ.

Solution to Additional Exercise C58

The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{A} = \begin{pmatrix} -2 & 7 \\ 1 & 4 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} -2-\lambda & 7 \\ 1 & 4-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(-2-\lambda)(4-\lambda) - 7 = 0,$$

which simplifies to

$$\lambda^2 - 2\lambda - 15 = (\lambda+3)(\lambda-5) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = -3$ and $\lambda = 5$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned} (-2-\lambda)x + 7y &= 0 \\ x + (4-\lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = -3}$ Both eigenvector equations become

$$x + 7y = 0.$$

Thus the eigenvectors corresponding to $\lambda = -3$ are the non-zero vectors for which $x = -7y$; that is, the vectors of the form

$$(-7k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 5}$ The eigenvector equations become

$$\begin{aligned} -7x + 7y &= 0 \\ x - y &= 0. \end{aligned}$$

These equations are equivalent to the single equation $x - y = 0$. Thus the eigenvectors corresponding to $\lambda = 5$ are the non-zero vectors for which $x = y$; that is, the vectors of the form

$$(k, k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of the linear transformation t are the non-zero vectors of the following forms:

$$\begin{aligned} &(-7k, k), \text{ corresponding to } \lambda = -3, \\ &(k, k), \text{ corresponding to } \lambda = 5. \end{aligned}$$

Solution to Additional Exercise C59

The matrix of t with respect to the standard basis for $\mathbb{R}^3$ is

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -2 & -2-\lambda & -2 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(3-\lambda)\begin{vmatrix} -2-\lambda & -2 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -2 \\ 1 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & -2-\lambda \\ 1 & 2 \end{vmatrix} = 0.$$

Simplifying this expression, we obtain

$$\begin{aligned} (3-\lambda)&(\lambda^2-4+4) - 2(2\lambda-4+2) + 2(-4+2+\lambda) \\ &= (3-\lambda)\lambda^2 - 2(2\lambda-2) + 2(\lambda-2) \\ &= -\lambda^3 + 3\lambda^2 - 2\lambda \\ &= -\lambda(\lambda^2 - 3\lambda + 2) \\ &= -\lambda(\lambda-2)(\lambda-1) = 0. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 2$, $\lambda = 1$ and $\lambda = 0$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned}(3-\lambda)x + {}&& 2y + {}&& 2z &= 0\\ -2x + {}&& (-2-\lambda)y - {}&& 2z &= 0\\ x + {}&& 2y + {}&& (2-\lambda)z &= 0.\end{aligned}$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned}x + 2y + 2z &= 0\\ -2x - 4y - 2z &= 0\\ x + 2y \phantom{{}+2z} &= 0.\end{aligned}$$

Subtracting the third equation from the first, we obtain $2z = 0$, which implies that $z = 0$. Substituting this into the equations gives the single equation $x + 2y = 0$. Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors for which $x = -2y$ and $z = 0$; that is, the vectors of the form

$$(-2k, k, 0), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 1}$ The eigenvector equations become

$$\begin{aligned}2x + 2y + 2z &= 0\\ -2x - 3y - 2z &= 0\\ x + 2y + z &= 0.\end{aligned}$$

Adding the first and second equations, we obtain $-y = 0$, which implies that $y = 0$. Substituting this into the equations gives the single equation $x + z = 0$. Thus the eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors for which $z = -x$ and $y = 0$; that is, the vectors of the form

$$(k, 0, -k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 0}$ The eigenvector equations become

$$\begin{aligned}3x + 2y + 2z &= 0\\ -2x - 2y - 2z &= 0\\ x + 2y + 2z &= 0.\end{aligned}$$

Adding the first and second equations, we obtain $x = 0$. Substituting this into the equations gives the single equation $y + z = 0$. Thus the eigenvectors corresponding to $\lambda = 0$ are the non-zero vectors for which $x = 0$ and $z = -y$; that is, the vectors of the form

$$(0, k, -k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of the linear transformation t are the non-zero vectors of the forms

$$\begin{aligned}&(-2k, k, 0), \text{ corresponding to } \lambda = 2,\\ &(k, 0, -k), \text{ corresponding to } \lambda = 1,\\ &(0, k, -k), \text{ corresponding to } \lambda = 0.\end{aligned}$$

Solution to Additional Exercise C60

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} a-\lambda & b\\ c & d-\lambda\end{vmatrix} = 0.$$

We expand this determinant and obtain

$$\begin{aligned}(a-\lambda)(d-\lambda) - bc &= \lambda^2 - (a+d)\lambda + ad - bc\\ &= 0.\end{aligned}$$

The result follows using the result from Subsection 1.4 of Unit A2 that the coefficient of λ is equal to minus the sum of the roots.

Alternatively, we can use the quadratic formula to obtain

$$\lambda_1 = \tfrac{1}{2}\left((a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right)$$

and

$$\lambda_2 = \tfrac{1}{2}\left((a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right).$$

The sum of these eigenvalues is

$$\tfrac{1}{2}(a+d) + \tfrac{1}{2}(a+d) = a + d,$$

as required.

Solution to Additional Exercise C61

The matrix of t with respect to the standard basis for $\mathbb{R}^3$ is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} -\lambda & 0 & -2\\ 1 & 2-\lambda & 1\\ 1 & 0 & 3-\lambda\end{vmatrix} = 0.$$

We expand this determinant and obtain

$$-\lambda\begin{vmatrix} 2-\lambda & 1\\ 0 & 3-\lambda\end{vmatrix} - 0 - 2\begin{vmatrix} 1 & 2-\lambda\\ 1 & 0\end{vmatrix} = 0.$$

Simplifying this expression, we obtain

$$\begin{aligned}-\lambda((2-\lambda)(3-\lambda) - 0) &- 2(0 - (2-\lambda))\\ &= (2-\lambda)(-\lambda(3-\lambda) + 2)\\ &= (2-\lambda)(\lambda^2 - 3\lambda + 2)\\ &= (2-\lambda)(\lambda-2)(\lambda-1) = 0.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 2$, $\lambda = 2$ and $\lambda = 1$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned} -\lambda x \qquad\quad - \qquad 2z &= 0 \\ x + (2-\lambda)y + \qquad z &= 0 \\ x \qquad\qquad + (3-\lambda)z &= 0. \end{aligned}$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned} -2x - 2z &= 0 \\ x + \ z &= 0 \\ x + \ z &= 0. \end{aligned}$$

These equations are equivalent to the single equation $x + z = 0$. There are no constraints on y. Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors for which $x = -z$; that is, the vectors of the form

$$(-k, l, k), \quad \text{where } k \text{ and } l \text{ are not both } 0.$$

The eigenspace $S(2)$ is the set of vectors

$$\{(-k, l, k) : k, l \in \mathbb{R}\}.$$

Any vector in $S(2)$ can be written as $k(-1,0,1) + l(0,1,0)$, so

$$\{(-1,0,1), (0,1,0)\}$$

is a basis for $S(2)$.

Thus $S(2)$ has dimension 2.

$\boxed{\lambda = 1}$ The eigenvector equations become

$$\begin{aligned} -x \qquad - 2z &= 0 \\ x + y + \ z &= 0 \\ x \qquad + 2z &= 0. \end{aligned}$$

The first and third equations imply that $x + 2z = 0$, so $x = -2z$. Substituting this into the second equation, we obtain $y - z = 0$, so $y = z$. Thus the eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors for which $x = -2z$ and $y = z$; that is, the vectors of the form

$$(-2k, k, k), \quad \text{where } k \neq 0.$$

The eigenspace $S(1)$ is the set of vectors

$$\{(-2k, k, k) : k \in \mathbb{R}\}.$$

Any vector in $S(1)$ can be written as $k(-2,1,1)$, so

$$\{(-2,1,1)\}$$

is a basis for $S(1)$.

Thus $S(1)$ has dimension 1.

Solution to Additional Exercise C62

(a) The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(b) First we find the eigenvalues of $\mathbf{A}$. The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(1-\lambda)^2 - 1 = 0,$$

which simplifies to

$$\lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 2$ and $\lambda = 0$.

Next we find the eigenvectors of $\mathbf{A}$. The eigenvector equations are

$$\begin{aligned} (1-\lambda)x + \qquad y &= 0 \\ x + (1-\lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned} -x + y &= 0 \\ x - y &= 0. \end{aligned}$$

These equations give $x = y$. Thus the eigenvectors corresponding to $\lambda = 2$ are the vectors of the form

$$(k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 0}$ The eigenvector equations become

$$\begin{aligned} x + y &= 0 \\ x + y &= 0. \end{aligned}$$

These equations give $x = -y$. Thus the eigenvectors corresponding to $\lambda = 0$ are the vectors of the form

$$(k, -k), \quad \text{where } k \neq 0.$$

The eigenvectors of $\mathbf{A}$ are the eigenvectors of t. It follows from Theorem C63 that we can form an eigenvector basis of t by taking one eigenvector corresponding to each eigenvalue. For example,

$$E = \{(1,1), (1,-1)\}.$$

(c) It follows from Theorem C59 that

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) It follows from Theorem C62 that $\mathbf{P}$ is the transition matrix from E to the standard basis for $\mathbb{R}^2$. So

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution to Additional Exercise C63

We use Strategy C20.

The eigenvalues of $\mathbf{A}$ are $\lambda = -3$ and $\lambda = 5$.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following forms:

$$(-7k, k), \text{ corresponding to } \lambda = -3,$$
$$(k, k), \text{ corresponding to } \lambda = 5.$$

It follows from Theorem C63 that we can form an eigenvector basis of $\mathbf{A}$ by taking one eigenvector corresponding to each eigenvalue. For example,

$$E = \{(7, -1), (1, 1)\}$$

is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 7 & 1 \\ -1 & 1 \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Solution to Additional Exercise C64

We use Strategy C20.

The eigenvalues of $\mathbf{A}$ are $\lambda = 2$, $\lambda = 1$ and $\lambda = 0$.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following forms:

$$(-2k, k, 0), \text{ corresponding to } \lambda = 2,$$
$$(k, 0, -k), \text{ corresponding to } \lambda = 1,$$
$$(0, k, -k), \text{ corresponding to } \lambda = 0.$$

It follows from Theorem C63 that we can form an eigenvector basis of $\mathbf{A}$ by taking one eigenvector corresponding to each of the three distinct eigenvalues. For example,

$$E = \{(-2, 1, 0), (1, 0, -1), (0, 1, -1)\}$$

is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution to Additional Exercise C65

The eigenvalues of $\mathbf{A}$ are $\lambda = 2$, $\lambda = 2$ and $\lambda = 1$.

The eigenspaces of $\mathbf{A}$ are

$$S(2) = \{(-k, l, k) : k, l \in \mathbb{R}\}$$
$$S(1) = \{(-2k, k, k) : k \in \mathbb{R}\}.$$

A basis for $S(2)$ is $\{(-1, 0, 1), (0, 1, 0)\}$ and a basis for $S(1)$ is $\{(-2, 1, 1)\}$. The set

$$E = \{(-1, 0, 1), (0, 1, 0), (-2, 1, 1)\}$$

contains three vectors, so it is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Additional Exercise C66

We use Strategy C22.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(5-\lambda)^2 - 1 = 0,$$

which simplifies to

$$\lambda^2 - 10\lambda + 24 = (\lambda - 6)(\lambda - 4) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 6$ and $\lambda = 4$.

The eigenvector equations are

$$\begin{aligned} (5-\lambda)x - y &= 0 \\ -x + (5-\lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = 6}$ The eigenvector equations become

$$\begin{aligned} -x - y &= 0 \\ -x - y &= 0. \end{aligned}$$

These equations are equivalent to the single equation $x + y = 0$; that is, $y = -x$. Thus the eigenvectors corresponding to $\lambda = 6$ are the non-zero vectors of the form $(k, -k)$.

An eigenvector of unit length corresponding to $\lambda = 6$ is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, -1)$.

$\boxed{\lambda = 4}$ The eigenvector equations become

$$\begin{aligned} x - y &= 0 \\ -x + y &= 0. \end{aligned}$$

These equations are equivalent to the single equation $x - y = 0$; that is, $y = x$. Thus the eigenvectors corresponding to $\lambda = 4$ are the non-zero vectors of the form (k, k).

An eigenvector of unit length corresponding to $\lambda = 4$ is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, 1)$.

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E = \left\{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}.$$

There are many possible solutions to each of Additional Exercises C66–C68, representing different orderings of the eigenvectors.

Solution to Additional Exercise C67

We use Strategy C22.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$-\lambda\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & -\lambda \end{vmatrix} + 0 = 0,$$

which simplifies to

$$\begin{aligned} -\lambda(-\lambda)^2 - (-\lambda) &= -\lambda(\lambda^2 - 1) \\ &= -\lambda(\lambda - 1)(\lambda + 1) = 0. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$.

The eigenvector equations are

$$\begin{aligned} -\lambda x + y &= 0 \\ x - \lambda y &= 0 \\ - \lambda z &= 0. \end{aligned}$$

$\boxed{\lambda = 1}$ The eigenvector equations become

$$\begin{aligned} -x + y &= 0 \\ x - y &= 0 \\ - z &= 0. \end{aligned}$$

The third equation implies that $z = 0$. The first and second equations imply that $y = x$. Thus the eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors of the form $(k, k, 0)$.

An eigenvector of unit length corresponding to $\lambda = 1$ is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(1, 1, 0)$.

$\boxed{\lambda = 0}$ The eigenvector equations become

$$\begin{aligned} y &= 0 \\ x &= 0 \\ 0z &= 0. \end{aligned}$$

The first two equations imply that $y = x = 0$. The third equation gives no constraints on z. Thus the eigenvectors corresponding to $\lambda = 0$ are the non-zero vectors of the form $(0, 0, k)$.

An eigenvector of unit length corresponding to $\lambda = 0$ is $(0, 0, 1)$.

$\boxed{\lambda = -1}$ The eigenvector equations become

$$\begin{aligned} x + y \quad &= 0 \\ x + y \quad &= 0 \\ z &= 0. \end{aligned}$$

The first two equations imply that $y = -x$. The third equation gives $z = 0$. Thus the eigenvectors corresponding to $\lambda = -1$ are the non-zero vectors of the form $(k, -k, 0)$.

An eigenvector of unit length corresponding to $\lambda = -1$ is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(1, -1, 0)$.

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0, 0, 1), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution to Additional Exercise C68

We use Strategy C22.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix} = 0.$$

We expand this determinant and obtain

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & -2-\lambda \end{vmatrix} + 4\begin{vmatrix} -4 & -2 \\ 2 & -2-\lambda \end{vmatrix} + 2\begin{vmatrix} -4 & 1-\lambda \\ 2 & -2 \end{vmatrix} = 0,$$

which simplifies to

$$\begin{aligned} &(1-\lambda)((1-\lambda)(-2-\lambda) - 4) + 4(-4(-2-\lambda) + 4) \\ &\qquad + 2(8 - 2(1-\lambda)) \\ &\quad = (1-\lambda)(\lambda^2 + \lambda - 6) + 4\,(4\lambda + 12) + 2\,(2\lambda + 6) \\ &\quad = (1-\lambda)(\lambda+3)(\lambda-2) + 16(\lambda+3) + 4(\lambda+3) \\ &\quad = (\lambda+3)(-\lambda^2 + 3\lambda + 18) \\ &\quad = -(\lambda+3)(\lambda-6)(\lambda+3) = 0. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 6$, $\lambda = -3$ and $\lambda = -3$.

The eigenvector equations are

$$\begin{aligned} (1-\lambda)x - \quad 4y + \quad 2z &= 0 \\ -4x + (1-\lambda)y - \quad 2z &= 0 \\ 2x - \quad 2y + (-2-\lambda)z &= 0. \end{aligned}$$

$\boxed{\lambda = 6}$ The eigenvector equations become

$$\begin{aligned} -5x - 4y + 2z &= 0 \\ -4x - 5y - 2z &= 0 \\ 2x - 2y - 8z &= 0. \end{aligned}$$

Adding the first and second equations gives $-9x - 9y = 0$, which implies that $y = -x$. Substituting this in the third equation gives $4x - 8z = 0$, which implies that $x = 2z$. Thus the eigenvectors corresponding to $\lambda = 6$ are the non-zero vectors of the form $(2k, -2k, k)$.

Now, $(2k, -2k, k) = k(2, -2, 1)$, so $\{(2, -2, 1)\}$ is a basis for $S(6)$.

An orthonormal basis for $S(6)$ is

$$\left\{\left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right)\right\} = \left\{\tfrac{1}{3}\,(2, -2, 1)\right\}.$$

$\boxed{\lambda = -3}$ The eigenvector equations become

$$\begin{aligned} 4x - 4y + 2z &= 0 \\ -4x + 4y - 2z &= 0 \\ 2x - 2y + \quad z &= 0. \end{aligned}$$

These equations are all equivalent to the single equation

$$2x - 2y + z = 0,$$

so $z = 2y - 2x$. Thus the eigenvectors corresponding to $\lambda = -3$ are the non-zero vectors of the form $(k, l, 2(l-k))$.

Now, $(k, l, 2(l-k)) = k(1, 0, -2) + l(0, 1, 2)$, so

$$\{(1, 0, -2), (0, 1, 2)\}$$

is a basis for $S(-3)$.

To find an orthogonal basis for $S(-3)$, we use the Gram–Schmidt orthogonalisation process.

Let the basis we seek be $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = (1, 0, -2)$. Let $\mathbf{w}_2 = (0, 1, 2)$.

Then

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\ &= (0, 1, 2) + \tfrac{4}{5}(1, 0, -2) \\ &= \left(\tfrac{4}{5}, 1, \tfrac{2}{5}\right). \end{aligned}$$

An orthonormal basis for $S(-3)$ is therefore

$$\left\{\left(\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}}\right),\left(\frac{4}{\sqrt{45}},\frac{5}{\sqrt{45}},\frac{2}{\sqrt{45}}\right)\right\}$$
$$=\left\{\frac{1}{\sqrt{5}}\left(1,0,-2\right),\frac{1}{3\sqrt{5}}\left(4,5,2\right)\right\}.$$

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$\left\{\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right),\left(\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}}\right),\left(\frac{4}{\sqrt{45}},\frac{5}{\sqrt{45}},\frac{2}{\sqrt{45}}\right)\right\}$$
$$=\left\{\frac{1}{3}\left(2,-2,1\right),\frac{1}{\sqrt{5}}\left(1,0,-2\right),\frac{1}{3\sqrt{5}}\left(4,5,2\right)\right\}.$$

We use the eigenvectors in this basis to form the columns of the transition matrix:

$$\mathbf{P}=\begin{pmatrix}\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}}\\ -\frac{2}{3} & 0 & \frac{5}{\sqrt{45}}\\ \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}}\end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P}=\mathbf{D}=\begin{pmatrix}6 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}.$$

Solution to Additional Exercise C69

If $\mathbf{A}$ is orthogonally diagonalisable, then there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{D}=\mathbf{P}^T\mathbf{A}\mathbf{P}$ is diagonal.

Now the matrix $\mathbf{D}$ is diagonal so $\mathbf{D}^T=\mathbf{D}$. Using the fact (from Subsection 5.4 of Unit C1) that for a pair of matrices $\mathbf{B}$ and $\mathbf{C}$ we have $(\mathbf{BC})^T=\mathbf{C}^T\mathbf{B}^T$, we get

$$\begin{aligned}\mathbf{D}=\mathbf{D}^T&=(\mathbf{P}^T\mathbf{A}\mathbf{P})^T\\&=\mathbf{P}^T\mathbf{A}^T(\mathbf{P}^T)^T\\&=\mathbf{P}^T\mathbf{A}^T\mathbf{P}.\end{aligned}$$

So $\mathbf{P}^T\mathbf{A}\mathbf{P}=\mathbf{P}^T\mathbf{A}^T\mathbf{P}$.

Multiplying on the right by $\mathbf{P}^T$ and the left by $\mathbf{P}$ we get $\mathbf{IAI}=\mathbf{I}\mathbf{A}^T\mathbf{I}$, and thus $\mathbf{A}=\mathbf{A}^T$ and hence the matrix $\mathbf{A}$ is symmetric.

Solution to Additional Exercise C70

(a) To verify that $\mathbf{A}$ is orthogonal, it is sufficient to show that $\mathbf{A}^T\mathbf{A}=\mathbf{I}$.

$$\mathbf{A}^T\mathbf{A}=\begin{pmatrix}\frac{2}{7} & -\frac{6}{7} & \frac{3}{7}\\ \frac{6}{7} & \frac{3}{7} & \frac{2}{7}\\ -\frac{3}{7} & \frac{2}{7} & \frac{6}{7}\end{pmatrix}\begin{pmatrix}\frac{2}{7} & \frac{6}{7} & -\frac{3}{7}\\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7}\\ \frac{3}{7} & \frac{2}{7} & \frac{6}{7}\end{pmatrix}$$
$$=\begin{pmatrix}\frac{49}{49} & 0 & 0\\ 0 & \frac{49}{49} & 0\\ 0 & 0 & \frac{49}{49}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I},$$

so $\mathbf{A}$ is orthogonal.

(b)

$$\mathbf{A}^{-1}=\mathbf{A}^T=\begin{pmatrix}\frac{2}{7} & -\frac{6}{7} & \frac{3}{7}\\ \frac{6}{7} & \frac{3}{7} & \frac{2}{7}\\ -\frac{3}{7} & \frac{2}{7} & \frac{6}{7}\end{pmatrix}.$$

(c) We evaluate the determinant of $\mathbf{A}$:

$$\begin{aligned}&\begin{vmatrix}\frac{2}{7} & \frac{6}{7} & -\frac{3}{7}\\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7}\\ \frac{3}{7} & \frac{2}{7} & \frac{6}{7}\end{vmatrix}\\&=\tfrac{2}{7}\begin{vmatrix}\frac{3}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{6}{7}\end{vmatrix}-\tfrac{6}{7}\begin{vmatrix}-\frac{6}{7} & \frac{2}{7}\\ \frac{3}{7} & \frac{6}{7}\end{vmatrix}-\tfrac{3}{7}\begin{vmatrix}-\frac{6}{7} & \frac{3}{7}\\ \frac{3}{7} & \frac{2}{7}\end{vmatrix}\\&=\tfrac{2}{7}\left(\tfrac{18}{49}-\tfrac{4}{49}\right)-\tfrac{6}{7}\left(-\tfrac{36}{49}-\tfrac{6}{49}\right)-\tfrac{3}{7}\left(-\tfrac{12}{49}-\tfrac{9}{49}\right)\\&=\tfrac{1}{343}(28+252+63)\\&=\tfrac{343}{343}=1.\end{aligned}$$

Therefore $\mathbf{A}$ represents a rotation of $\mathbb{R}^3$.

Solution to Additional Exercise C71

We use Strategy C24.

1. Introduce matrices.

   We have

   $$\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\quad\text{and}\quad\mathbf{J}=\begin{pmatrix}-4\\ -12\end{pmatrix}.$$

2. Align the axes.

   The matrix is already in diagonal form. (The axes of the conic are parallel to the x-axis and y-axis of $\mathbb{R}^2$.)

3. Translate the origin.

   We write the equation as

   $$(x^2-4x)-2(y^2+6y)-18=0.$$

   Completing the squares in the equation, we obtain

   $$(x-2)^2-4-2(y+3)^2+18-18=0.$$

Substituting

$$x' = x - 2, \quad \text{and} \quad y' = y + 3$$

in this equation and simplifying, we obtain

$$(x')^2 - 2(y')^2 - 4 = 0.$$

The equation of the conic in standard form is

$$\frac{(x')^2}{4} - \frac{(y')^2}{2} = 1.$$

The conic is a hyperbola.

There are many possible solutions to each of Additional Exercises C71–C75, representing different orderings of the eigenvectors.

Solution to Additional Exercise C72

We use Strategy C24.

1. Introduce matrices.

   We have

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2. Align the axes.

   In Additional Exercise C66 you showed that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix},$$

   where

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

   There are no linear x or y terms, so

$$\begin{pmatrix} f & g \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

   The equation of the conic is now

$$6(x')^2 + 4(y')^2 - 1 = 0.$$

3. Translate the origin.

   No translation is required: the conic is centred at the origin, since there are no linear x or y terms.

The equation of the conic in standard form is

$$6(x')^2 + 4(y')^2 = 1.$$

The conic is an ellipse.

Solution to Additional Exercise C73

We use Strategy C25.

1. Introduce matrices.

   We have

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} -6 \\ 10 \\ 1 \end{pmatrix}.$$

2. Align the axes.

   In Additional Exercise C67 you showed that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

   where

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.$$

   So

$$\begin{aligned} \begin{pmatrix} f & g & h \end{pmatrix} &= \begin{pmatrix} -6 & 10 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{\sqrt{2}} & 1 & -\frac{16}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 2\sqrt{2} & 1 & -8\sqrt{2} \end{pmatrix}. \end{aligned}$$

   The equation of the quadric is now

$$(x')^2 - (z')^2 + 2\sqrt{2}x' + y' - 8\sqrt{2}z' - 30 = 0.$$

3. Translate the origin.

   We write the equation as

$$\left((x')^2 + 2\sqrt{2}x'\right) + y' - \left((z')^2 + 8\sqrt{2}z'\right) - 30 = 0.$$

   Completing the squares in this equation, we obtain

$$(x' + \sqrt{2})^2 - 2 + y' - (z' + 4\sqrt{2})^2 + 32 - 30 = 0.$$

   Substituting

$$x'' = x' + \sqrt{2}, \quad y'' = y', \quad z'' = z' + 4\sqrt{2}$$

   in this equation and simplifying, we obtain

$$(x'')^2 + y'' - (z'')^2 = 0.$$

The equation of the quadric in standard form is

$$y'' = (z'')^2 - (x'')^2.$$

This is the equation of a hyperbolic paraboloid.

Solution to Additional Exercise C74

We use Strategy C25.

1. Introduce matrices.

   We have

   $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

2. Align the axes.

   The matrix is already in diagonal form. (The axes of the quadric are parallel to the x-axis, y-axis and z-axis of $\mathbb{R}^3$.)

3. Translate the origin.

   We write the equation as

   $$(x^2 + x) + y^2 - z = 0.$$

   Completing the square in the equation, we obtain

   $$\left(x + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + y^2 - z = 0.$$

   Substituting

   $$x' = x + \tfrac{1}{2}, \quad y' = y, \quad z' = z + \tfrac{1}{4}$$

   in this equation and simplifying, we obtain

   $$(x')^2 + (y')^2 - z' = 0.$$

The equation of the quadric in standard form is

$$z' = (x')^2 + (y')^2.$$

This is the equation of an elliptic paraboloid.

Solution to Additional Exercise C75

We use Strategy C25.

1. Introduce matrices.

   We have

   $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. Align the axes.

   This matrix $\mathbf{A}$ is the same as the matrix $\mathbf{A}$ in Additional Exercise C73, so we have

   $$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

   where

   $$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.$$

   So

   $$\begin{aligned} \begin{pmatrix} f & g & h \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

   The equation of the quadric is now

   $$(x')^2 - (z')^2 + y' = 0.$$

3. Translate the origin.

   No translation is required: the quadric is centred at the origin, since there are no linear x' or z' terms.

The equation of the quadric in standard form is

$$y' = -(x')^2 + (z')^2.$$

This is the equation of a hyperbolic paraboloid.